Tensor renormalization group approach to quantum fields on a lattice

Shinichiro Akiyama [a), b)]

[a)] Endowed Chair for Quantum Software, University of Tokyo

[b)] Institute for Physics of Intelligence, University of Tokyo

2022.5.26 @ ipi seminar, University of Tokyo

Contents

1. Background & Motivation

2. Introduction to the TRG approach

   - How to derive tensor network representations
   - TRG algorithms
   - Status of the TRG approach

3. Application to (3+1)d QFTs on a lattice

   - Complex ϕ^4 theory → scalar field
   - NJL model → fermionic (Grassmann) field
   - $\mathbb{Z}_2$ gauge-Higgs model → gauge field

4. Summary

Background & Motivation

Motivation of the field theory on a lattice

✔ The field theory (FT) is a fundamental tool to describe the high-energy physics

→ We usually consider (would like to solve) the path integral of a theory

✔ Once we consider the FT on a lattice, we can regard the path integral just as a multiple integral

$$\int [\mathrm{d}\phi] \cdots \quad \rightarrow \quad \int \prod_{n \in \Lambda_d} \mathrm{d}\phi(n) \cdots$$

Numerical approaches give us chances to understand non-perturbative physics

Standard numerical approach for FT on a lattice

✔ Monte Carlo (MC) simulation (Stochastic numerical approach)

→ based on the probabilistic interpretation of the given Boltzmann weight

→ faces a serious difficulty when $P \sim \mathrm{e}^{-S[\phi]}$ takes negative or complex value

・**Sign problem (there are regimes where the MC does not work)**

・**Fermions (the Grassmann numbers) must be integrated out in advance**

Research Motivation

✔ There are many systems suffering from the sign problem

QCD at finite density / real-time evolution / SUSY / …

✔ Many unrevealed aspects must be remained

→ Thermodynamic limit (or zero-temperature limit) is almost inaccessible w/ the standard MC approach

✔ **We need a numerical methodology which can give us an insight for these aspects**

We would like to consider the TRG approach from these perspectives

Advantages of the TRG approach

✔ Tensor renormalization group (TRG) is a deterministic numerical method based on the idea of the real-space renormalization group

- **No sign problem**
- **The computational cost scales logarithmically w. r. t. the system size**
- **Direct evaluation of the Grassmann integrals**
- **Direct evaluation of the path integral**

✔ **Applicability to the higher-dimensional systems**

TRG is a kind of tensor-network method and the application of the TRG to the higher-dimensional systems has recently made remarkable progress

Lagrangian (TRG) approach: Meurice+, arXiv:2010.06539, SA+, arXiv:2111:04240

Hamiltonian (TNS) approach: Bañuls-Cichy, Rep. Prog. Phys. 83(2020)024401

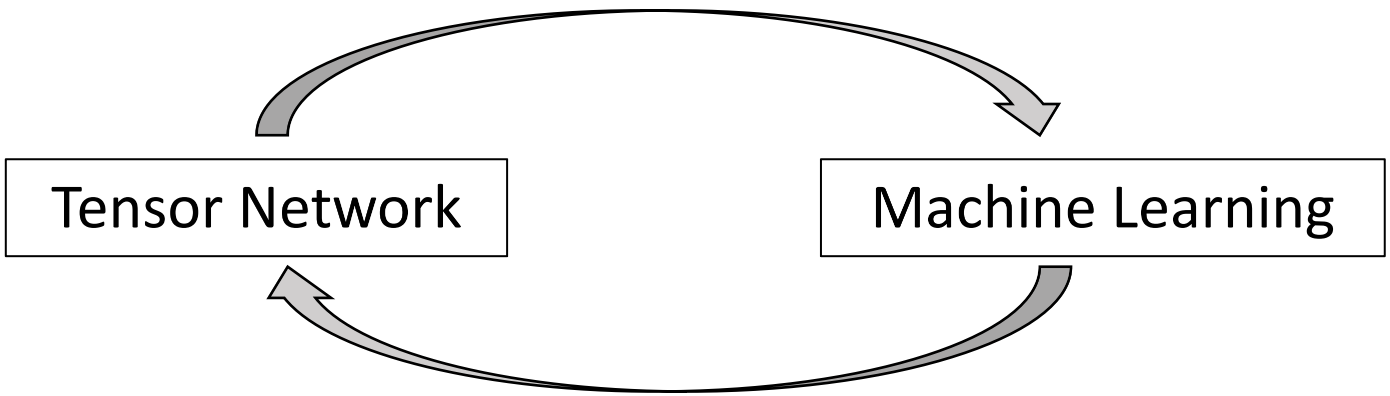
Cf. Tensor network & Machine Learning

Ex) Supervised image classification of the MNIST handwritten digits

Stoudenmire-Schwab, Advances in Neural Information Processing Systems 29(2016)4799

Martyn+, arXiv:2007.06082[quant-ph]

Matrix Product State (MPS)
Tensor Network State (TNS)



Back propagation method

Liao+, PRX9(2019)031041

Ex) A new way to accurately compute higher-order derivatives of the free energy

Introduction to the TRG approach

Tensor renormalization group approach

Procedure of the TRG

1) Write down the target function X defined on lattice as a tensor contraction (tensor network)

ex. Partition function, Path integral, …

2) Approximately perform the tensor contraction with a TRG

1) **TN representation for $\boldsymbol{X}$** : $(\#\text{ of tensors in TN}) = (\#\text{ of lattice sites})$

$$X \rightarrow \Sigma_{abcd\cdots} T_{aiw\cdots} T_{bjx\cdots} T_{cky\cdots} T_{dlz\cdots} \cdots$$

2) **TRG** : Block-spin trans. for T to reduce # of tensors in TN

$$\approx \Sigma_{a'b'c'd'\cdots} {T'}_{a'i'w'\cdots} {T'}_{b'j'x'\cdots} {T'}_{c'k'y'\cdots} {T'}_{d'l'z'\cdots} \cdots$$

TN rep. for 2d Ising model w/ PBC

Decompose nearest-neighbor interactions

$$Z = \Sigma_{\{\sigma=\pm1\}} \Pi_{n,\mu} \exp\left[\beta J \sigma_n \sigma_{n+\hat{\mu}}\right] \quad \Rightarrow \quad Z = \mathrm{Tr}\left[\Pi_n T_{x_n y_n x'_n y'_n}\right]$$

$T_{x_n y_n x'_n y'_n}$ specifies the details of the model

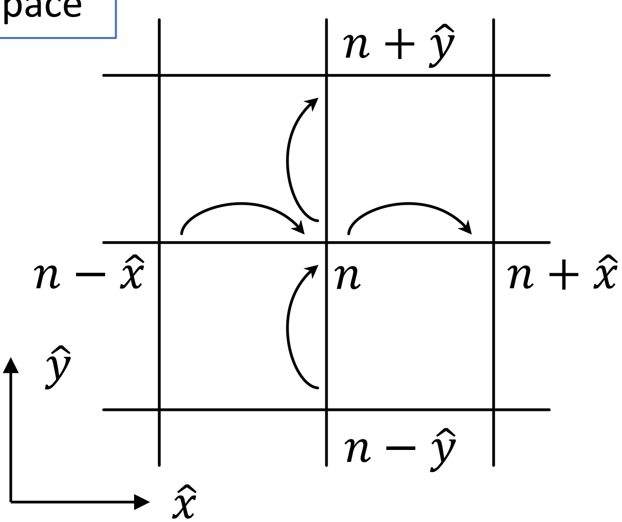
$$\exp\left[\beta J \sigma_n \sigma_{n+\hat{\mu}}\right] = \sum_{l_n} \sqrt{\lambda_{l_n}} U(\sigma_n, l_n) \sqrt{\lambda_{l_n}} U\left(\sigma_{n+\hat{\mu}}, l_n\right) = \sum_{l_n} W(\sigma_n, l_n) W\left(\sigma_{n+\hat{\mu}}, l_n\right)$$

$$W(a,b) \coloneqq \sqrt{\lambda_b} U(a,b)$$

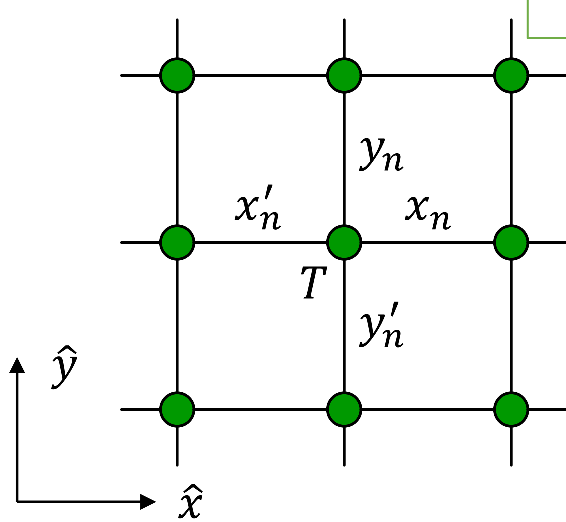
$$T_{x_n y_n x'_n y'_n} \coloneqq \sum_{\sigma_n=\pm1} W(\sigma_n, x_n) W(\sigma_n, y_n) W(\sigma_n, x'_n) W(\sigma_n, y'_n)$$

$$x'_n \coloneqq x_{n-\hat{x}},\ y'_n \coloneqq y_{n-\hat{y}}$$

Real Space

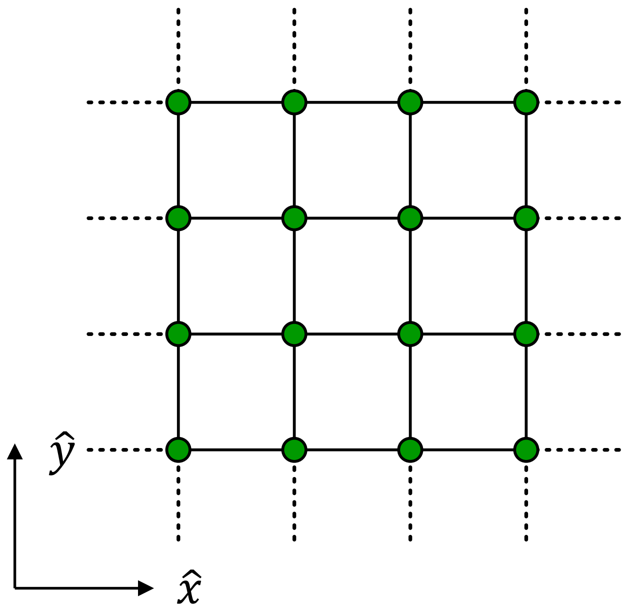


TN rep. for Z



Basic concept of TRG algorithm

We cannot perform the contractions in TN rep. exactly (too many d. o. f.)



Idea of real-space renormalization group

Iterate a simple transformation **w/ approximation** and we can investigate thermodynamic properties

+

Information compression w/ the Singular Value Decomposition (SVD)

$$A_{ij} = \Sigma_k U_{ik}\sigma_k V_{jk} \approx \Sigma_{k=1}^{D} U_{ik}\sigma_k V_{jk}$$

w/ $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$

(A: $m \times n$ matrix, U: $m \times m$ unitary, V: $n \times n$ unitary)

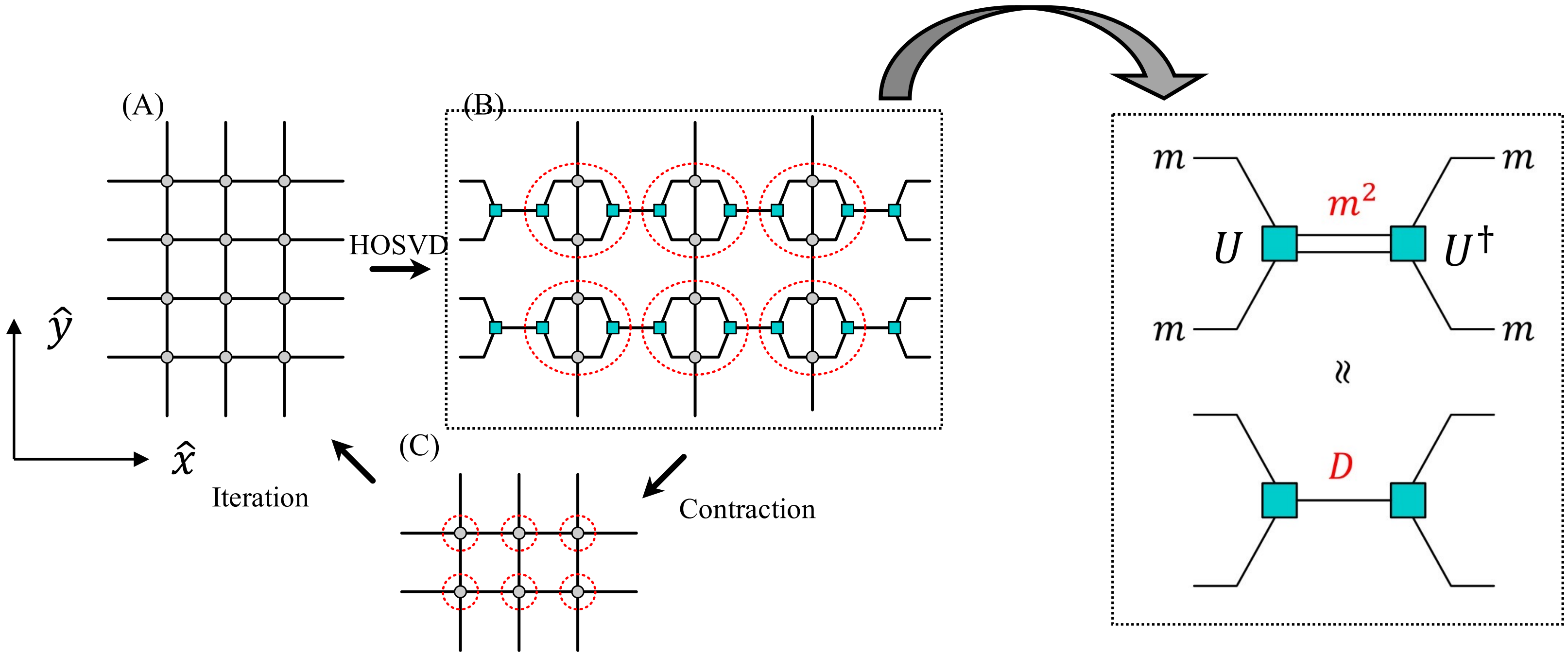
↓

TRG employs the SVD to reduce d. o. f. and perform the tensor contraction approximately

Higher-order TRG (= TRG w/ isometry insertion)

Xie et al, PRB86(2012)045139

✔ Applicable to any d-dimensional lattice



Sequential coarse-graining along with each direction

# of tensors are reduced to half

D: bond dimension

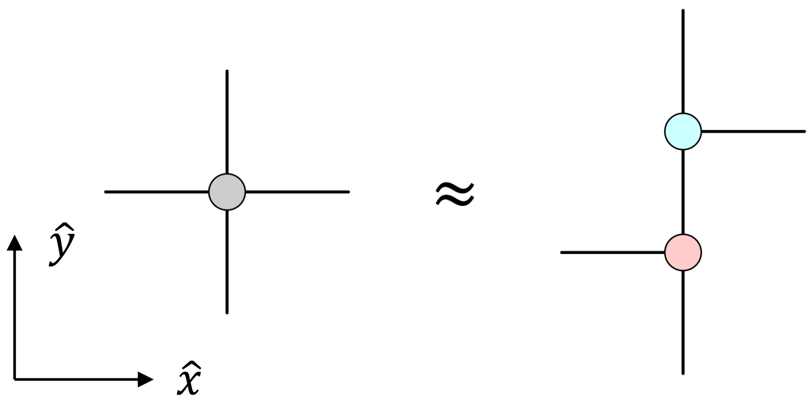
Maximal size of tensors in the TRG algorithm is characterized by D

Anisotropic TRG (= TRG w/ indirect SVD)

Adachi-Okubo-Todo, PRB102(2020)054432

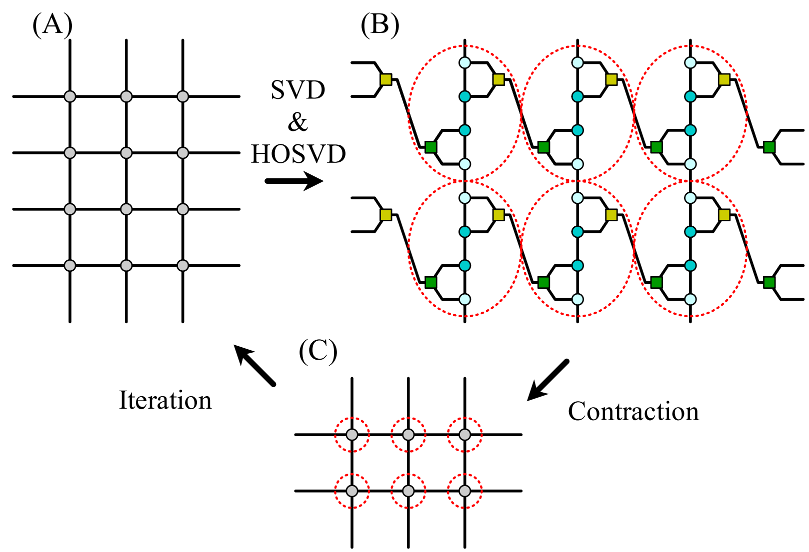
✔ Applicable to any d-dimensional lattice

✔ Accuracy with the fixed computational time is improved compared with the HOTRG, which is a conventional algorithm to the higher-dimensional systems



$$T_{xyx'y'} \approx \sum_{i=1}^{D} A_{xyi} B_{x'y'i}$$

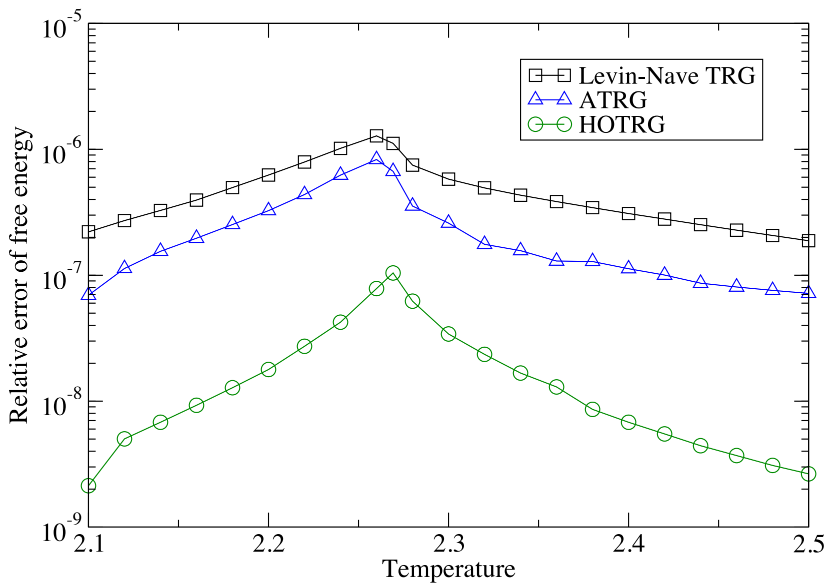
ATRG considers the block-spin transformation within lower-rank tensors (Memory: $O(D^{2d}) \to O(D^{d+1})$)



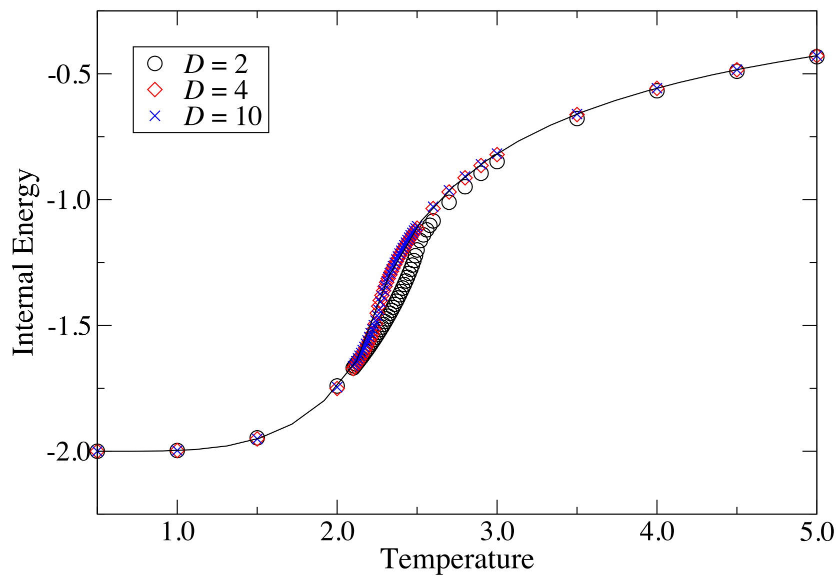
# of tensors are reduced to half

Benchmarking w/ 2d Ising model (1/2)

Comparison of three types of TRG
w/ $D = 24$



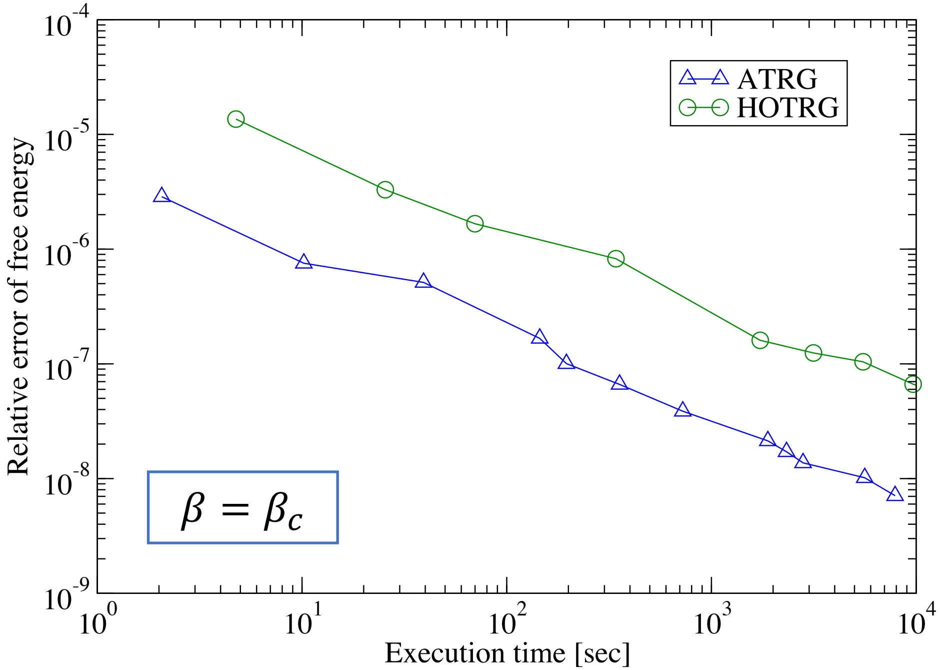
HOTRG calculation



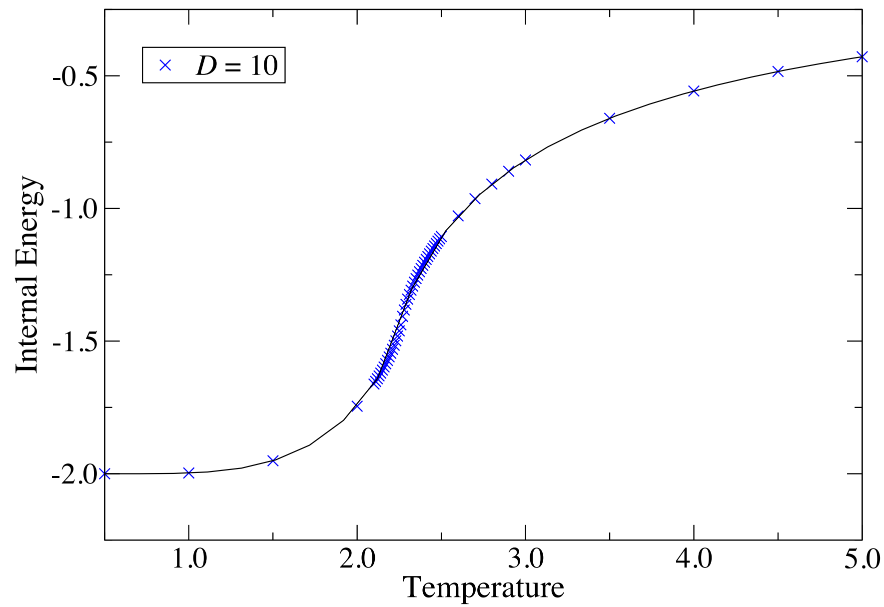
HOTRG & ATRG improve the accuracy of the original (LN-)TRG at the same D
The exact solution is well reproduced

Benchmarking w/ 2d Ising model (2/2)

Relative error vs execution time



ATRG calculation

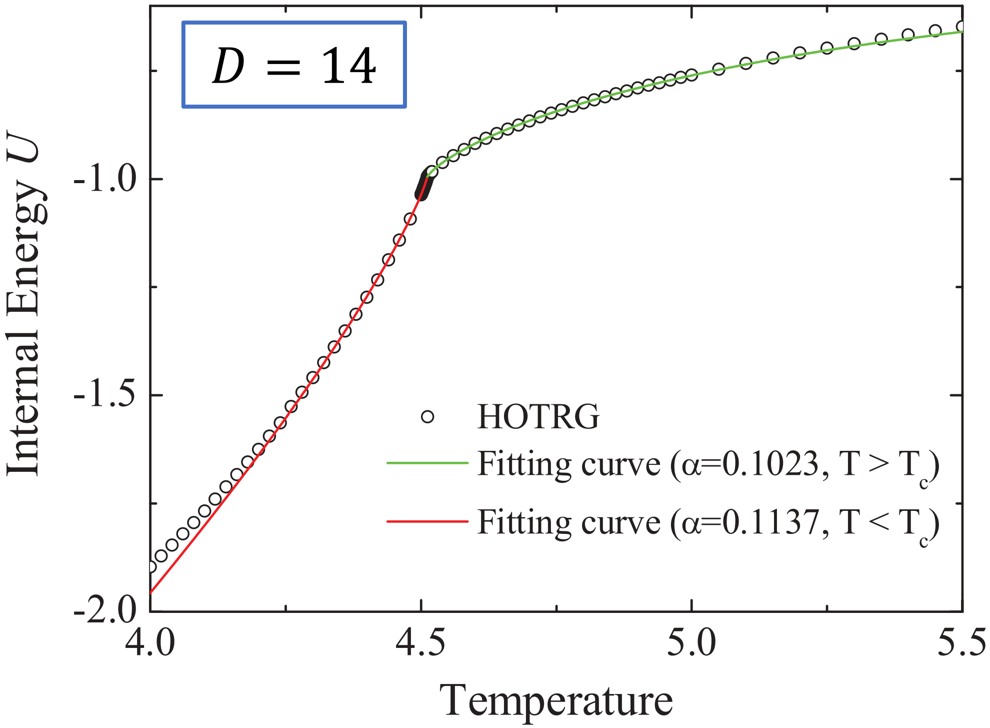


ATRG shows better performance than the HOTRG at the same execution time

	2d ATRG	2d HOTRG	LN-TRG
Memory	$O(D^3)$	$O(D^4)$	$O(D^4)$
Time	$O(D^5)$	$O(D^7)$	$O(D^6)$

Example: 3d Ising model w/ HOTRG

Xie et al, PRB86(2012)045139



Critical point

Method	T_c
HOTRG (D = 16, from U)	4.511544
HOTRG (D = 16, from M)	4.511546
Monte Carlo[37]	4.511523
Monte Carlo[38]	4.511525
Monte Carlo[39]	4.511516
Monte Carlo[35]	4.511528
Series expansion[40]	4.511536
CTMRG[12]	4.5788
TPVA[13]	4.5704
CTMRG[14]	4.5393
TPVA[16]	4.554
Algebraic variation[41]	4.547

Good agreement with
the Monte Carlo results

Current status of TRG in the higher-dimensional systems

Algorithm	Cost	Applications to 3d	Applications to 4d
HOTRG Xie et al, PRB86(2012)045139	$D^{4d-1}\ln L$	Ising Xie+, Potts model Wang+, free Wilson fermion Sakai+, $\mathbb{Z}_2$ gauge theory Dittirich+, Kuramashi-Yoshimura	○Ising model SA+, Staggered fermion w/strongly coupled U(N) Milde+
Anisotropic TRG (ATRG) Adachi-Okubo-Todo, PRB102(2020)054432	$D^{2d+1}\ln L$	Ising model Adachi+, SU(2) gauge Kuwahara-Tsuchiya, ○Real ϕ^4 theory SA+, ○Hubbard model SA-Kuramashi ○$\mathbb{Z}_2$ gauge-Higgs SA-Kuramashi	○Complex ϕ^4 theory SA+, ○NJL model SA+, ○Real ϕ^4 theory SA+ ○$\mathbb{Z}_2$ gauge-Higgs SA-Kuramashi
Triad RG Kadoh-Nakayama, arXiv:1912.02414	$D^{d+3}\ln L$	Ising model Kadoh-Nakayama, O(2) model Bloch+, $\mathbb{Z}_3$ (extended) clock model Bloch+ Potts models Raghav G. Jha	-

D: bond dimension, L: linear system size, d: spacetime dimension

Application to (3+1)d QFTs on a lattice

(3+1)d complex ϕ^4 theory at finite density

S. A., D. Kadoh, Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP09(2020)177

✔ Efficiency of the TRG in the scalar theory w/ the sign problem

✔ Discretization (Regularization) of the continuous bosonic dof

(3+1)d Nambu—Jona-Lasinio model at finite density

S. A., Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP01(2021)121

✔ Efficiency of the TRG in the fermion theory w/ the sign problem

✔ We can directly manipulate the Grassmann integral w/ the TRG

(3+1)d $\mathbb{Z}_2$ gauge-Higgs model

If time allows…

S. A. and Y. Kuramashi, JHEP05(2022)102

✔ Efficiency of the TRG in the higher-dimensional LGT

Application to (3+1)d QFTs on a lattice

(3+1)d complex ϕ^4 theory at finite density

S. A., D. Kadoh, Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP09(2020)177

✔ Efficiency of the TRG in the scalar theory w/ the sign problem

✔ Discretization (Regularization) of the continuous bosonic dof

(3+1)d Nambu—Jona-Lasinio model at finite density

S. A., Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP01(2021)121

✔ Efficiency of the TRG in the fermion theory w/ the sign problem

✔ We can directly manipulate the Grassmann integral w/ the TRG

(3+1)d $\mathbb{Z}_2$ gauge-Higgs model

If time allows…

S. A. and Y. Kuramashi, JHEP05(2022)102

✔ Efficiency of the TRG in the higher-dimensional LGT

Complex ϕ^4 theory at finite density

✔ a typical system with the sign problem

✔ the Silver Blaze phenomenon

→ thermodynamic observables at zero temperature are independent of μ up to μ_c

✔ a testbed for the methods intended to overcome the sign problem

- Complex Langevin method
  Aarts, PRL102(2009)131601
- Thimble approach
  Cristoforetti et al, PRD88(2013)051501
  Fujii et al, JHEP10(2013)147
- World-line representation
  Gattringer-Kloiber, NPB869(2013)56-73

Discretization of the bosonic dof

✔ Typical system w/ the sign problem at finite chemical potential μ

$$S = \Sigma_{n\in\Lambda_d}[-\Sigma_{\nu=1}^{d}\left(\mathrm{e}^{\mu\delta_{\nu,d}}\phi_n^*\phi_{n+\nu} + \mathrm{e}^{-\mu\delta_{\nu,d}}\phi_n\phi_{n+\nu}^*\right) + (2d+m^2)|\phi_n|^2 + \lambda|\phi_n|^4]$$

✔ Employing the Gauss quadrature rule, we regularize (discretize) $\phi_n \in \mathbb{C}$ to obtain TN rep.

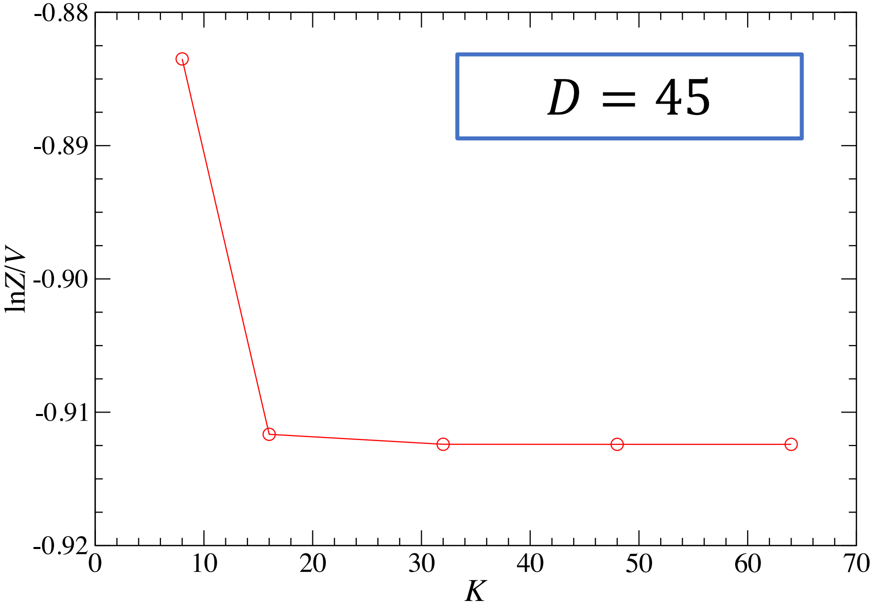
Polar-coordinate description: $\phi_n = r_n\mathrm{e}^{\mathrm{i}\pi s_n}$

Continuous d. o. f.		**Discrete d. o. f.**	**Quadrature rule**
$r_n \in [0,\infty]$	$\longrightarrow$	$\alpha_n \in \mathbb{Z}$	Gauss-Laguerre : $\int_0^\infty \mathrm{d}r_n\, \mathrm{e}^{-r_n} f(r_n) \approx \sum_{\alpha_n=0}^{K} w_{\alpha_n} f(r_{\alpha_n})$
$s_n \in [-1,1]$	$\longrightarrow$	$\beta_n \in \mathbb{Z}$	Gauss-Legendre: $\int_{-1}^{1} \mathrm{d}s_n f(s_n) \approx \sum_{\beta_n=0}^{K} u_{\beta_n} f(s_{\beta_n})$

Algorithmic-parameters dependence

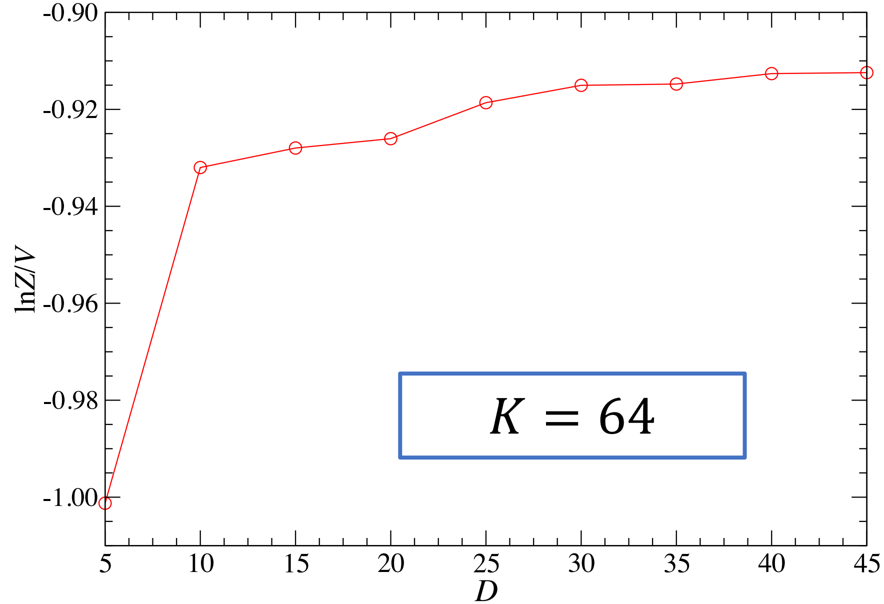
with $m = 0.1, \lambda = 1, \mu = 0.6, L = 1024$

Polynomial order
in the Gauss quadrature



little K dependence beyond $K \sim 30$

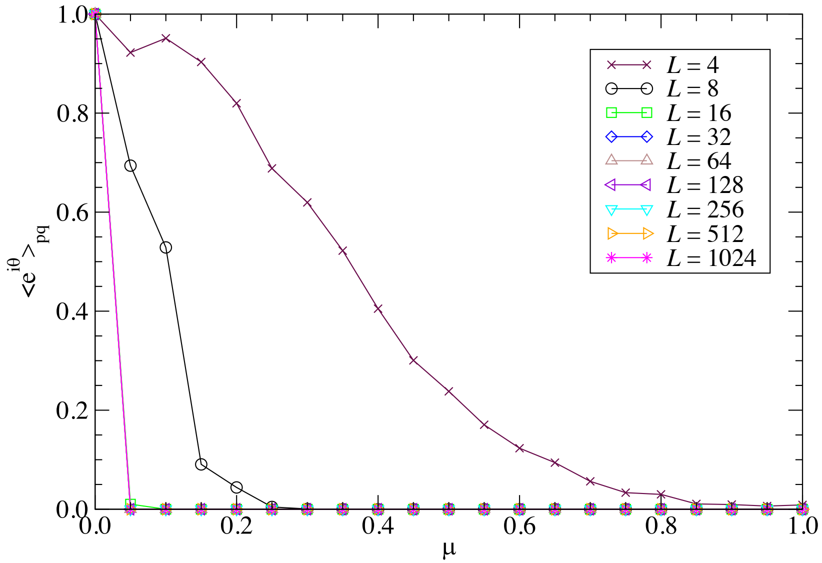
Bond dimension in the ATRG



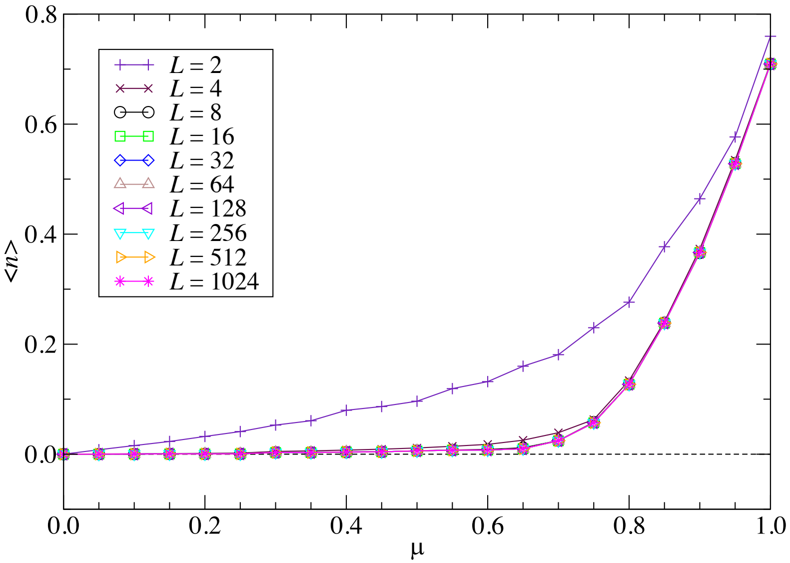
converging around $D \sim 40$

Silver Blaze Phenomenon at finite density

Average phase factor



Number density



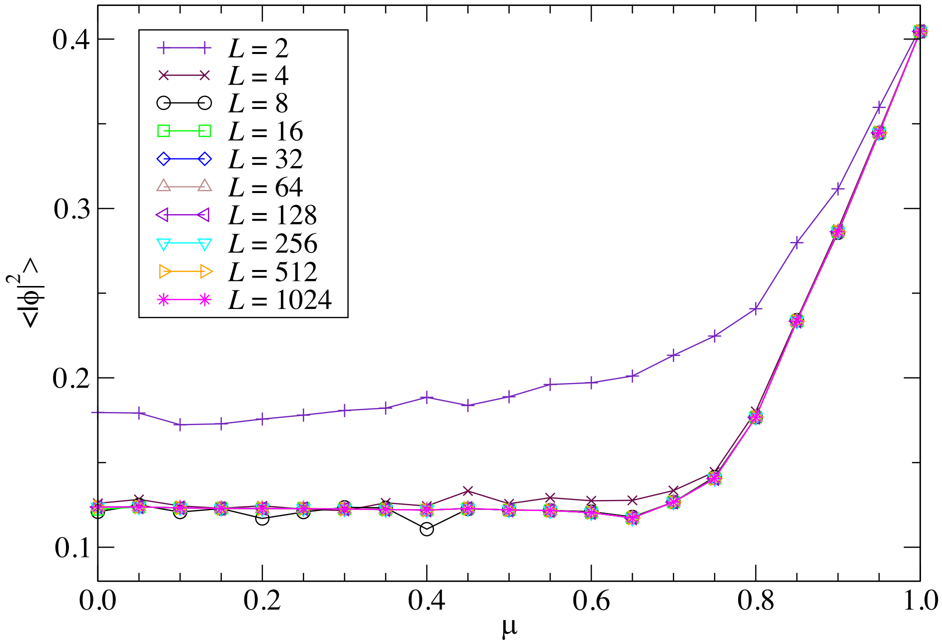
Resulting $\langle n \rangle$ is qualitatively not bad even in the region with $\langle \mathrm{e}^{\mathrm{i}\theta} \rangle_{\mathrm{pq}} \sim 0$.

$\langle n \rangle$ stays around 0 up to $\mu \approx 0.65$ and shows the rapid increase with $\mu \gtrsim 0.65$.

$\rightarrow$ This is a typical feature of the Silver Blaze phenomenon

$\langle|\phi|^2\rangle$: a discussion of the validity of numerical results

with $m = 0.1, \lambda = 1, K = 64, D = 45$



$\langle|\boldsymbol{\phi}|^2\rangle \approx \mathbf{0.125}$ over $0 \lesssim \mu \lesssim 0.6$

Mean-field estimation

$$4\sinh^2\frac{\mu_c^{\mathrm{MF}}}{2} = m^2 + 4\lambda\langle|\phi|^2\rangle_{\mu=0}$$

Aarts, JHEP05(2009)052

$\downarrow$

$\boldsymbol{\mu_c^{\mathrm{MF}} \approx 0.70}$

Location of $\boldsymbol{\mu_c}$ in the current ATRG calculations seems reasonable

Application to (3+1)d QFTs on a lattice

(3+1)d complex ϕ^4 theory at finite density

S. A., D. Kadoh, Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP09(2020)177

✔ Efficiency of the TRG in the scalar theory w/ the sign problem

✔ Discretization (Regularization) of the continuous bosonic dof

(3+1)d Nambu—Jona-Lasinio model at finite density

S. A., Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP01(2021)121

✔ Efficiency of the TRG in the fermion theory w/ the sign problem

✔ We can directly manipulate the Grassmann integral w/ the TRG

(3+1)d $\mathbb{Z}_2$ gauge-Higgs model

If time allows…

S. A. and Y. Kuramashi, JHEP05(2022)102

✔ Efficiency of the TRG in the higher-dimensional LGT

Expected phase diagram of the NJL model

- ✔ **Effective theory of QCD**

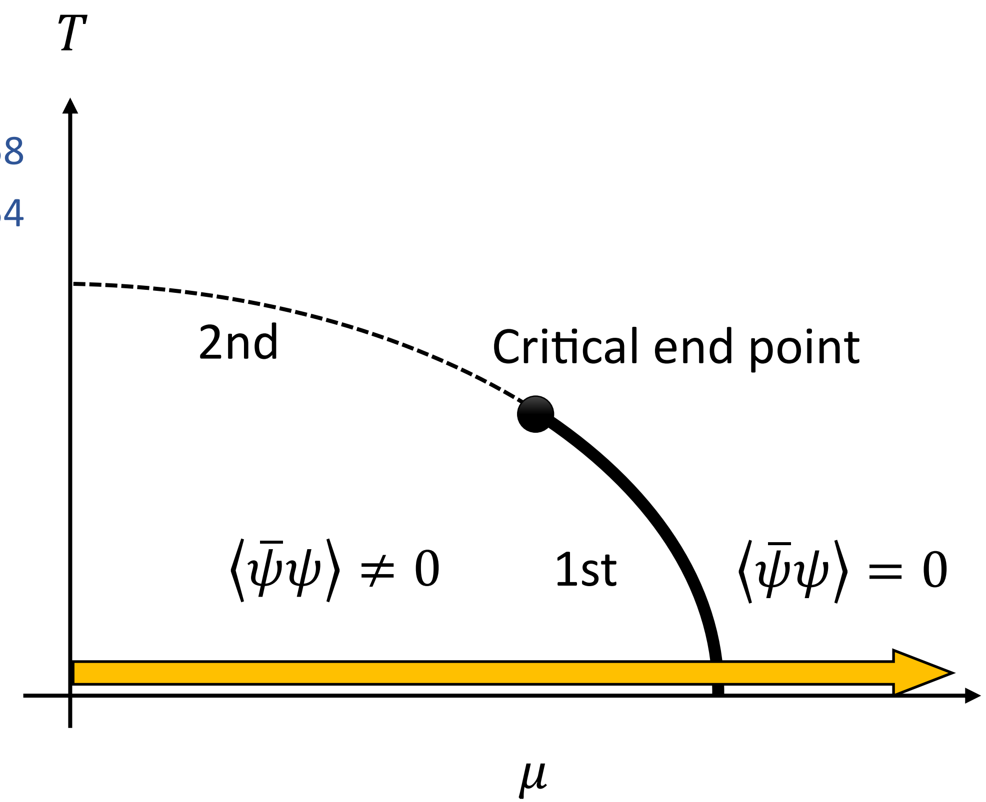
  Nambu–Jona-Lasinio, PRD122(1961)345-358

  Nambu–Jona-Lasinio, PRD124(1961)246-254

- ✔ **Chiral restoration is expected in cold & dense region**

  Asakawa-Yazaki, NPA504(1989)668-684

- ✔ **Severe sign problem in cold & dense region**



We apply the Tensor Renormalization Group (TRG) approach to investigate the 1st order chiral phase transition in cold & dense region

NJL model at finite density

✔ w/ the Kogut-Susskind fermion

→ Single-component Grassmann variables w/o the Dirac structure

→ Staggered sign function $\eta_\nu(n) = (-1)^{n_1+\cdots+n_{\nu-1}}$ with $\eta_1(n) = 1$

✔ μ : chemical potential

$$S_{\mathrm{lat}} = \frac{1}{2} a^3 \Sigma_{n\in\Lambda} \Sigma_{\nu=1}^{4} \eta_\nu(n) \left[\mathrm{e}^{\mu a \delta_{\nu,4}} \bar{\chi}(n) \chi(n+\hat{\nu}) - \mathrm{e}^{-\mu a \delta_{\nu,4}} \bar{\chi}(n+\hat{\nu}) \chi(n) \right]$$

$$+ m a^4 \Sigma_{n\in\Lambda} \bar{\chi}(n)\chi(n) - g_0 a^4 \Sigma_{n\in\Lambda} \Sigma_{\nu=1}^{4} \bar{\chi}(n)\chi(n)\bar{\chi}(n+\hat{\nu})\chi(n+\hat{\nu})$$

(This formulation follows Lee-Shrock, PRL59(1987)14)

✔ Continuous chiral symmetry for vanishing m :

$$\chi(n) \to \mathrm{e}^{\mathrm{i}\alpha\epsilon(n)} \chi(n), \qquad \bar{\chi}(n) \to \bar{\chi}(n) \mathrm{e}^{\mathrm{i}\alpha\epsilon(n)}$$

w/ $\alpha \in \mathbb{R}$ and $\epsilon(n) = (-1)^{n_1+n_2+n_3+n_4}$

Auxiliary fermion fields to derive the TN rep.

SA-Kadoh, JHEP10(2021)188

Decompose hopping structures via

$$\mathrm{e}^{A\bar{\boldsymbol{\psi}}_n \boldsymbol{\psi}_{n+\mu}} = \left(\int\int \mathrm{d}\bar{\eta}_n \mathrm{d}\eta_n \mathrm{e}^{-\bar{\eta}_n \eta_n}\right) \exp\left[-\sqrt{A}\bar{\boldsymbol{\psi}}_{\boldsymbol{n}} \eta_n + \sqrt{A}\bar{\eta}_n \boldsymbol{\psi}_{\boldsymbol{n}+\boldsymbol{\mu}}\right]$$

Original Z	**TN rep for Z**
$\int$ over $\{\bar{\psi}, \psi\}$	$\int$ with the weight $\mathrm{e}^{-\bar{\eta}\eta}$ over $\{\bar{\eta}, \eta\}$

Integrating out $\{\bar{\psi}, \psi\}$

Original Grassmann numbers are manifestly converted into the Grassmann numbers (= auxiliary fermion fields)

TN rep. for a simple fermionic model

$$Z=\int[\mathrm{d}\psi\mathrm{d}\bar{\psi}]\Pi_{n,\mu}\mathrm{e}^{A\bar{\psi}_{n+\hat{\mu}}\psi_n+B\bar{\psi}_n\psi_{n+\hat{\mu}}} \quad\Rightarrow\quad Z=\text{“Tr”}\left[\Pi_n\mathcal{T}_{\Psi_x(n)\Psi_y(n)\bar{\Psi}_y(n)\bar{\Psi}_x(n)}\right]$$

“Tr” denotes the weighted Grassmann integrals

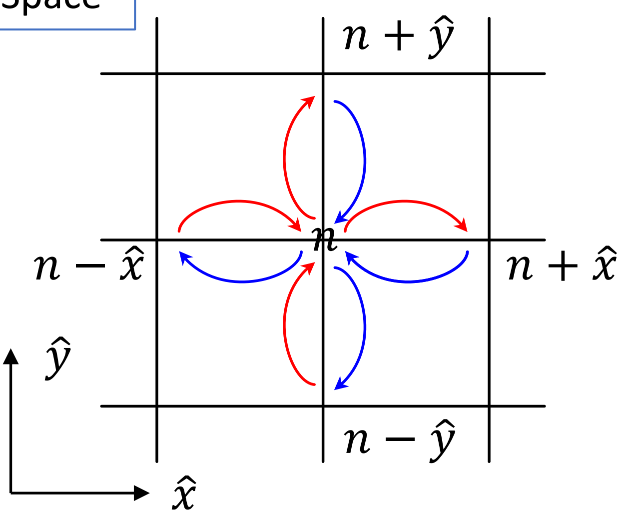
$$\mathrm{e}^{A\bar{\psi}_{n+\hat{\mu}}\psi_n}=\left(\int\int\mathrm{d}\bar{\eta}_\mu(n)\mathrm{d}\eta_\mu(n)\mathrm{e}^{-\bar{\eta}_\mu(n)\eta_\mu(n)}\right)\exp\left[-\sqrt{A}\bar{\psi}_{n+\hat{\mu}}\bar{\eta}_\mu(n)+\sqrt{A}\eta_\mu(n)\psi_n\right]$$

$$\mathrm{e}^{B\bar{\psi}_n\psi_{n+\hat{\mu}}}=\left(\int\int\mathrm{d}\bar{\zeta}_\mu(n)\mathrm{d}\zeta_\mu(n)\mathrm{e}^{-\bar{\zeta}_\mu(n)\zeta_\mu(n)}\right)\exp\left[\sqrt{B}\bar{\psi}_n\zeta_\mu(n)+\sqrt{B}\bar{\zeta}_\mu(n)\psi_{n+\hat{\mu}}\right]$$

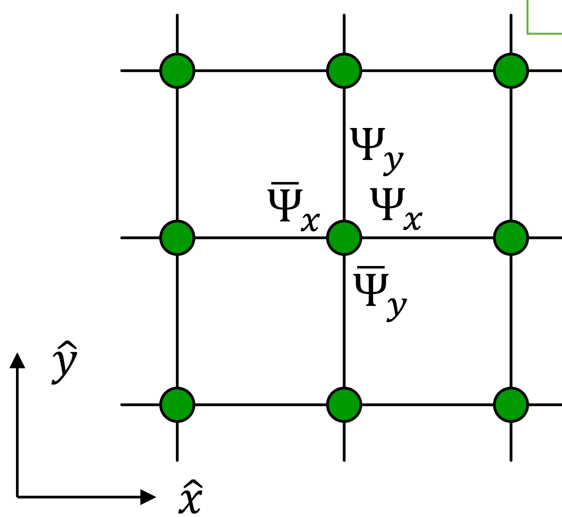
$$\mathcal{T}_{\Psi_x(n)\Psi_y(n)\bar{\Psi}_y(n)\bar{\Psi}_x(n)}$$
$$:=\int\mathrm{d}\psi_n\mathrm{d}\bar{\psi}_n\Pi_{\mu=x,y}\exp\left[\sqrt{A}\eta_\mu(n)\psi_n+\sqrt{B}\bar{\psi}_n\zeta_\mu(n)+\sqrt{B}\bar{\zeta}_\mu(n-\hat{\mu})\psi_n-\sqrt{A}\bar{\psi}_n\bar{\eta}_\mu(n-\hat{\mu})\right]$$

$\Psi_\mu(n)=\left(\eta_\mu(n),\zeta_\mu(n)\right)$ and $\bar{\Psi}_\mu(n)=\left(\bar{\eta}_\mu(n-\hat{\mu}),\bar{\zeta}_\mu(n-\hat{\mu})\right)$

Real Space



Tensor Network



Correspondence btw tensors and the G tensors

G tensor is a multi-linear combination like $\mathcal{T}_{\alpha\beta\bar{\beta}\bar{\alpha}} = \Sigma_{iji'j'} \boldsymbol{T}_{\boldsymbol{iji'j'}} \alpha^{i} \beta^{j} \bar{\beta}^{j'} \bar{\alpha}^{i'}$

	Tensor	Grassmann tensor
index	integer	Grassmann number
contraction	$\Sigma_i \cdots$	$\int\int \mathrm{d}\bar{\eta}\mathrm{d}\eta \mathrm{e}^{-\bar{\eta}\eta} \cdots$

Any TRG algorithm can be easily extended to investigate the Grassmann path integral

(We can easily encode the Grassmann algebra in a coefficient tensor $T_{iji'j'}$)

SA-Kadoh, JHEP10(2021)188, SA+, arXiv:2111:04240

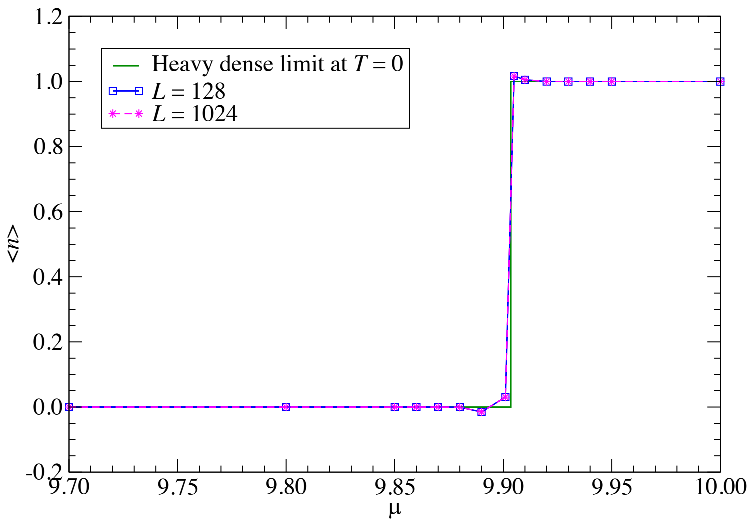
Heavy dense limit as a benchmark

with $m = 10^4, g_0 = 32, D = 30$

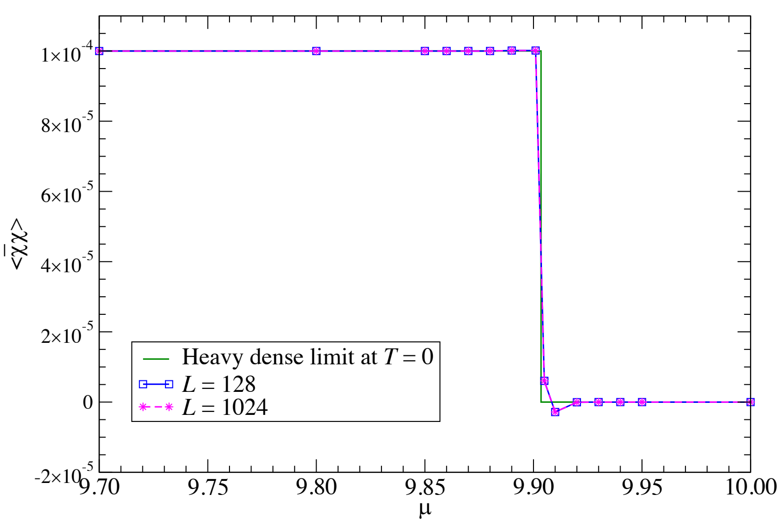
This limit allows us to compare numerical results w/ exact analytical ones

cf. Powlowski-Zielinski, PRD87(2013)094509

Number density



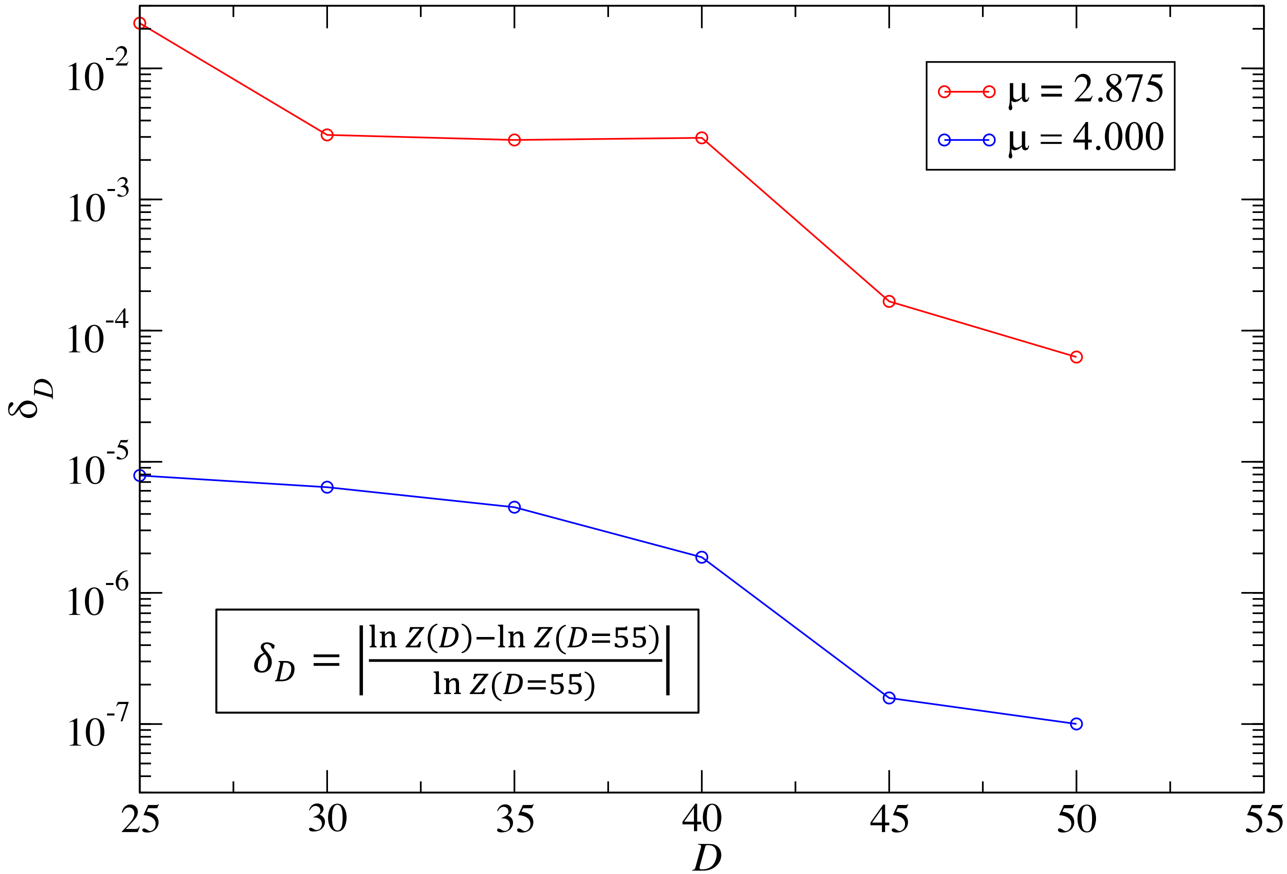
Fermion condensate



GATRG well reproduces the analytical results, including the location of $\mu_c = \ln(2m) = 9.903$

Converging behavior in bond dimension

with $m = 0.01, g_0 = 32, L = 1024$

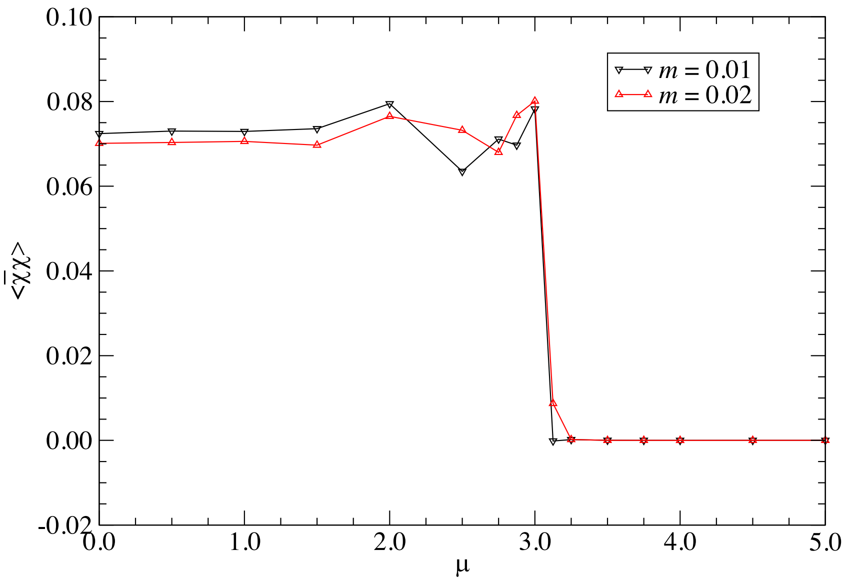


$\delta_D \lesssim 10^{-4}$ has been achieved up to $D = 55$ at $\mu \approx \mu_c$

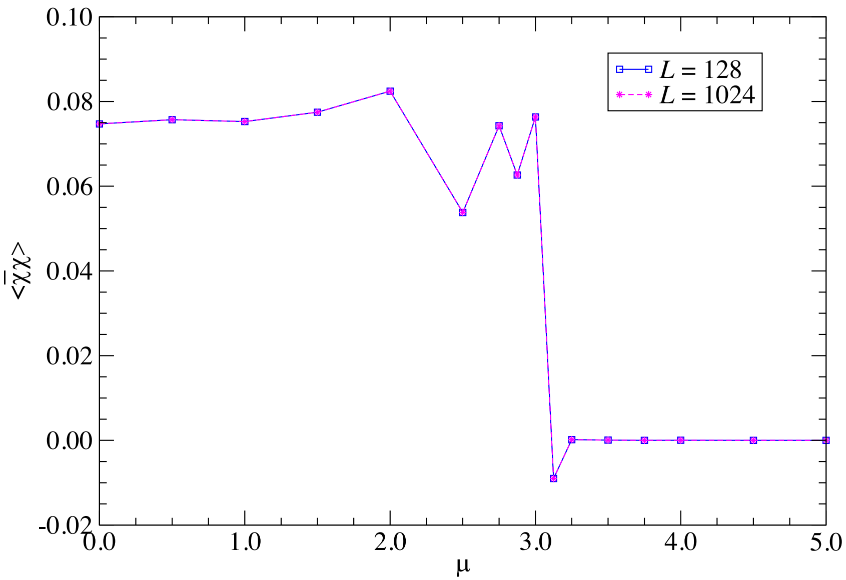
Chiral condensate

with $g_0 = 32, D = 55$

$L = 1024^4$



$m \to 0$



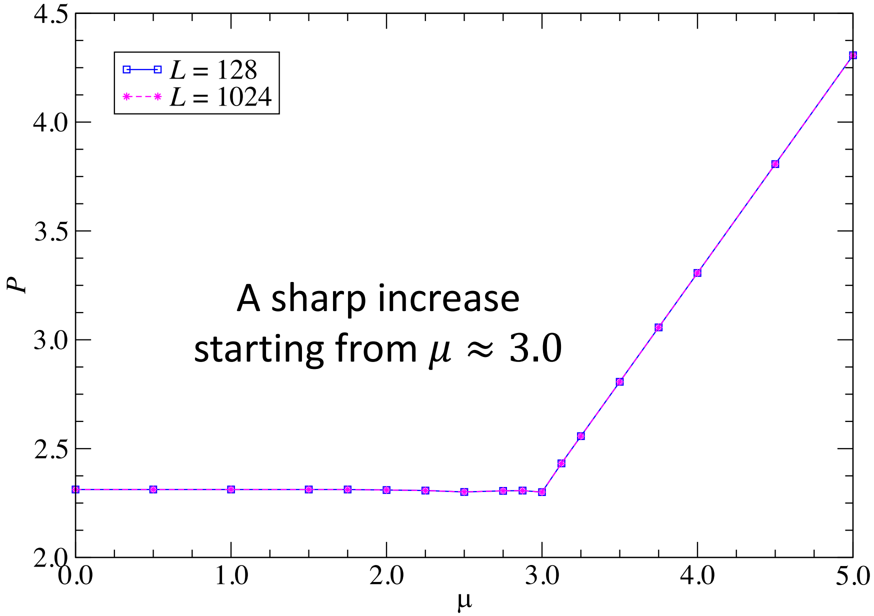
Chiral symmetry is restored in the region with $\mu \gtrsim 3.0$

A discontinuity at $\mu \approx 3.0$ indicates the 1st order transition

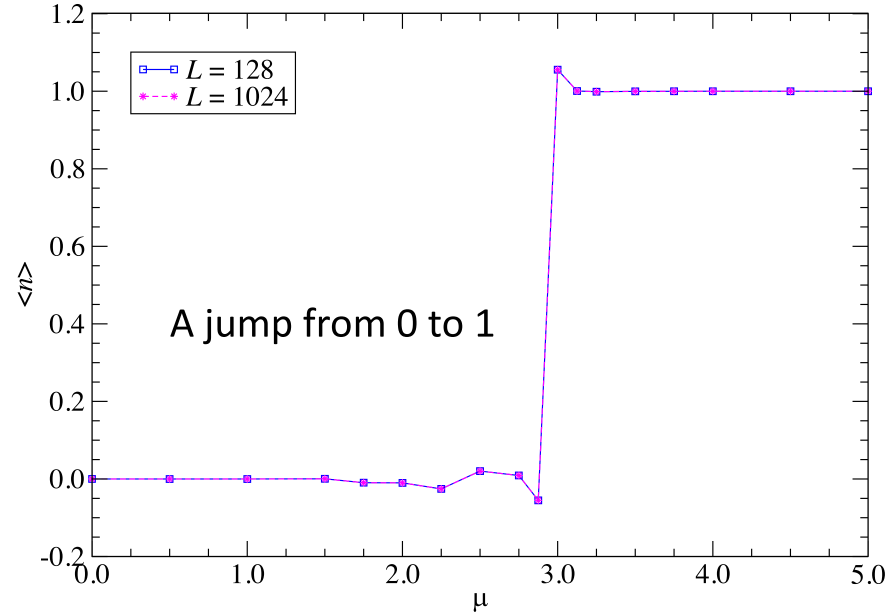
Ingredients of the equation of state

with $m = 0.01, g_0 = 32, D = 55$

Pressure
(~ Thermodynamic potential)



Number density



Chiral phase transition in cold & dense region is 1st order

Summary

- The TRG approach does not suffer from the sign problem and allows us to investigate the thermodynamic limit
- No difficulty to apply the approach to scalar and fermion fields on a lattice
- **We made the first application of the TRG to (3+1)d QFTs on a lattice, including scalar, fermion, and gauge theories.**
- Steady progress of the TRG approach has been made toward numerical research on the regimes which are inaccessible with the standard MC approach, even in higher dimensions
- **How about a higher-dimensional gauge theory w/ fermionic matter, like QED?**

Application to (3+1)d QFTs on a lattice

(3+1)d complex ϕ^4 theory at finite density

S. A., D. Kadoh, Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP09(2020)177

✔ Efficiency of the TRG in the scalar theory w/ the sign problem

✔ Discretization (Regularization) of the continuous bosonic dof

(3+1)d Nambu—Jona-Lasinio model at finite density

S. A., Y. Kuramashi, T. Yamashita and Y. Yoshimura, JHEP01(2021)121

✔ Efficiency of the TRG in the fermion theory w/ the sign problem

✔ We can directly manipulate the Grassmann integral w/ the TRG

(3+1)d $\mathbb{Z}_2$ gauge-Higgs model

If time allows…

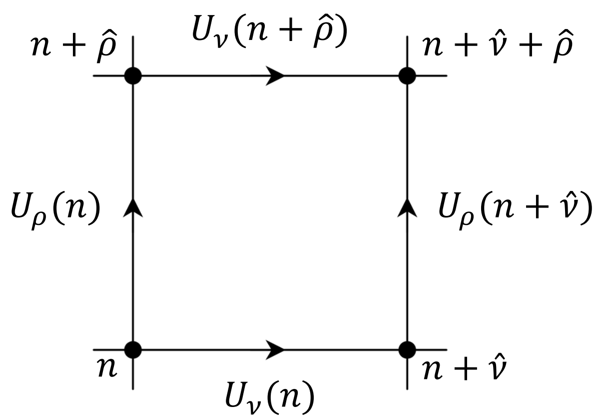
S. A. and Y. Kuramashi, JHEP05(2022)102

✔ Efficiency of the TRG in the higher-dimensional LGT

$\mathbb{Z}_2$ gauge-Higgs model in the unitary gauge

✔ Action of the $(d+1)$-dimensional $\mathbb{Z}_2$ gauge-Higgs model

$$S = -\beta \sum_n \sum_{\nu>\rho} U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu(n+\hat{\rho}) U_\rho(n)$$
$$-\eta \sum_n \sum_\nu \left[\mathrm{e}^{\mu \delta_{\nu,d+1}} \sigma(n) U_\nu(n) \sigma(n+\hat{\nu}) + \mathrm{e}^{-\mu \delta_{\nu,d+1}} \sigma(n) U_\nu(n-\hat{\nu}) \sigma(n-\hat{\nu}) \right]$$



$U_\nu(n)(\in \mathbb{Z}_2)$: link variable (gauge field)
$\sigma(n)(\in \mathbb{Z}_2)$: matter field

✔ **Choosing the unitary gauge, all the matter fields are eliminated**

$$\sigma(n) U_\nu(n) \sigma(n+\hat{\nu}) \mapsto U_\nu(n)$$

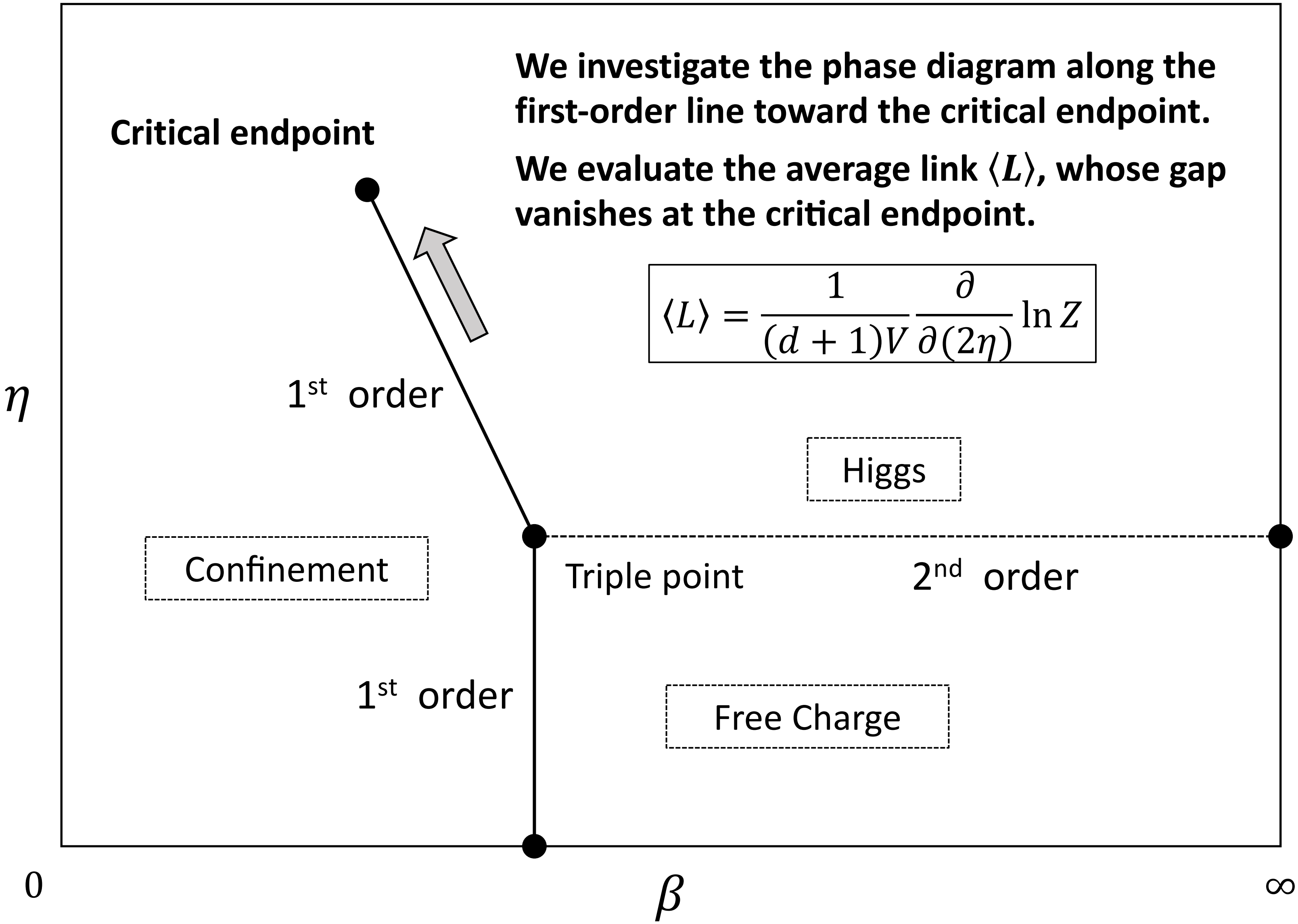
$$\boxed{S = -\beta \sum_n \sum_{\nu>\rho} U_\nu(n) U_\rho(n+\hat{\nu}) U_\nu(n+\hat{\rho}) U_\rho(n) - 2\eta \sum_n \sum_\nu \cosh(\mu \delta_{\nu,d+1}) U_\nu(n)}$$

Phase diagram of the (3+1)d $\mathbb{Z}_2$-Higgs model

We investigate the phase diagram along the first-order line toward the critical endpoint.

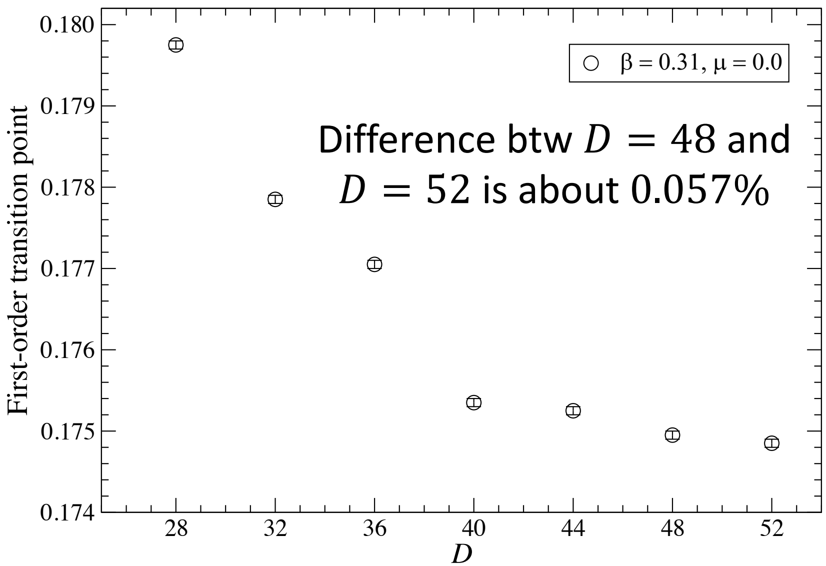
We evaluate the average link $\langle L \rangle$, whose gap vanishes at the critical endpoint.

$$\langle L \rangle = \frac{1}{(d+1)V} \frac{\partial}{\partial(2\eta)} \ln Z$$

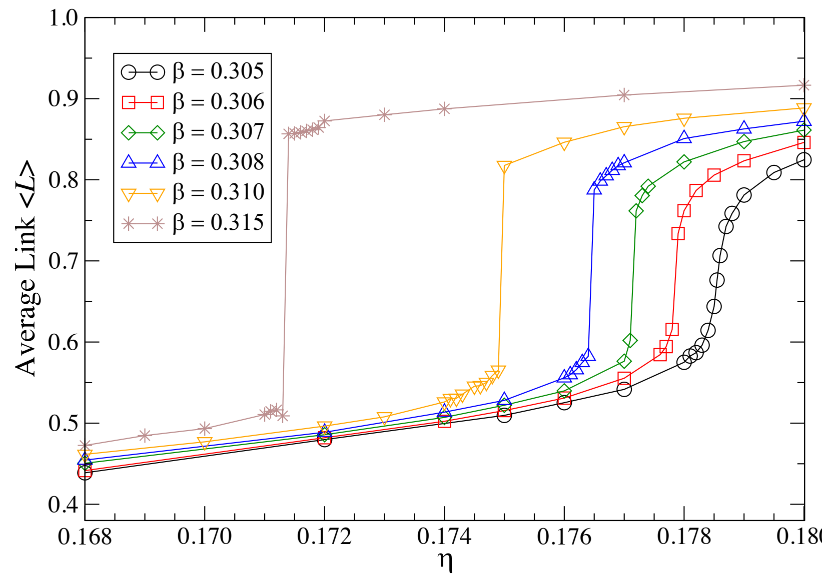


(3+1)D model at vanishing density

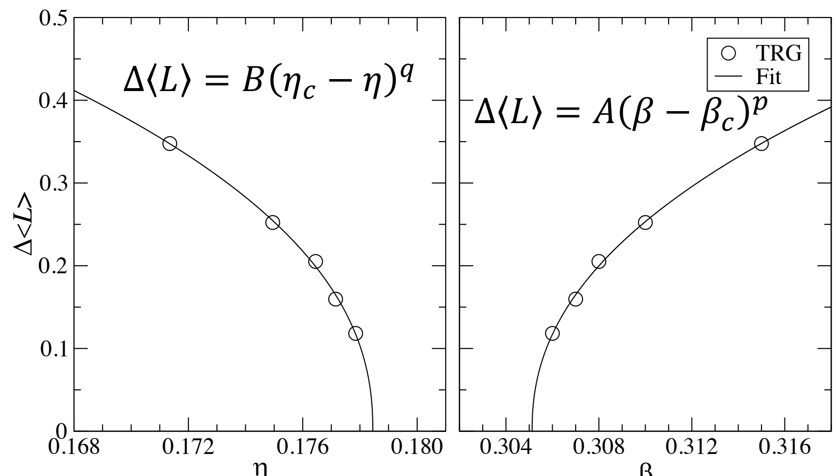
with $D \leq 52,\ \ \eta_+ - \eta_- = O(10^{-4})$



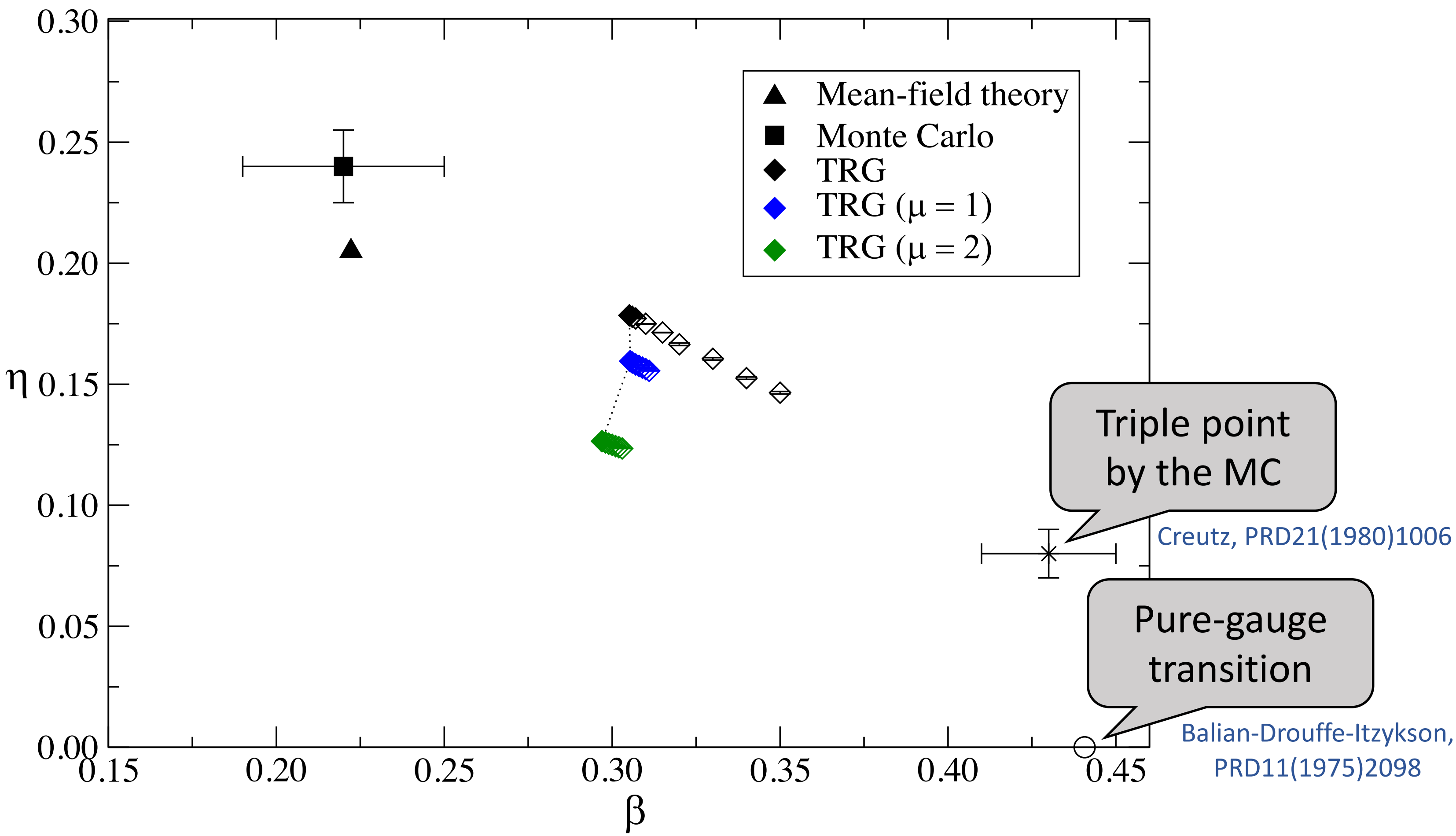
Bond dimension



Mean-field Brezin-Drouffe, NPB200(1982)93	$(\beta_c, \eta_c) = (0.22, 0.205)$
MC Creutz, PRD21(1980)1006	$(\beta_c, \eta_c) = (0.22(3), 0.24(2))$
TRG w/ $D = 52$ this work	$(\beta_c, \eta_c) = (0.3051(2), 0.1784(2))$



Current status of the phase diagram near the CEP



It seems that TRG and MC share a similar first-order line at $\mu = 0$

A deviation about the location of the CEP