13 Quantization of Dirac field

Canonical Quantization The Lagrangian for the Dirac field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = \psi^\dagger(i\partial_0)\psi + \cdots \tag{13.1}$$

implies that the canonical momentum for the Dirac field is,

$$\psi_\alpha = \partial(\mathcal{L})/\partial(\dot{\psi}_\alpha) = i\psi^*_\alpha \tag{13.2}$$

The canonical (equal-time) anti-commutation relation reads,

$$\{\psi_\alpha(\vec{x}), \pi_\beta(\vec{y})\} = i\delta(\vec{x}-\vec{y})\delta_{\alpha\beta} \quad \rightarrow \quad \left\{\psi_\alpha(\vec{x}), \psi^*_\beta(\vec{y})\right\} = \delta(\vec{x}-\vec{y})\delta_{\alpha\beta} \tag{13.3}$$

with $\{\psi, \psi\} = \{\psi^*, \psi^*\} = 0$.

Mode expansion Mode expansion is defined by the normalized solutions of Dirac equation $u_r(\vec{p}), v_r(\vec{p})$ as,

$$\begin{aligned}
\psi(x) &= \psi^{(+)}(x) + \psi^{(-)}(x) \\
\psi^{(+)}(x) &= \int d^3x \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} \sum_r c_r(\vec{p}) u_r(\vec{p}) e^{-ipx} \\
\psi^{(-)}(x) &= \int d^3x \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} \sum_r d^\dagger_r(\vec{p}) v_r(\vec{p}) e^{ipx}
\end{aligned} \tag{13.4}$$

$$\begin{aligned}
\bar{\psi}(x) &= \bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x) \\
\bar{\psi}^{(-)}(x) &= \int d^3x \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} \sum_r c^\dagger_r(\vec{p}) \bar{u}_r(\vec{p}) e^{ipx} \\
\bar{\psi}^{(+)}(x) &= \int d^3x \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} \sum_r d_r(\vec{p}) \bar{v}_r(\vec{p}) e^{-ipx}
\end{aligned} \tag{13.5}$$

The canonical anti-commutation relation is equivalent to

$$\left\{c_r(\vec{p}), c^\dagger_r(\vec{q})\right\} = \left\{d_r(\vec{p}), d^\dagger_r(\vec{q})\right\} = \delta_{rs}\delta(\vec{p}-\vec{q})\,, \tag{13.6}$$

and all the other anti-commutator vanishes. In order to prove the equivalence, we need relations,

$$\sum_r u_r(\vec{p})_\alpha \bar{u}_r(\vec{q})_\beta = (\gamma^\mu p_\mu + m)_{\alpha\beta}, \quad \sum_r v_r(\vec{p})_\alpha \bar{v} v_r(\vec{q})_\beta = (\gamma^\mu p_\mu - m)_{\alpha\beta}, \tag{13.7}$$

which we have proved previously.

Energy, Momentum, Charge The conserved quantities for the fermion system is conveniently expressed in terms of number operators,

$$N_r(\vec{p}) = c_r^\dagger(\vec{p})c_r(\vec{p}), \quad \bar{N}_r(\vec{p}) = d_r^\dagger(\vec{p})d_r(\vec{p}) \tag{13.8}$$

which counts the number of fermion and anti-fermions. The total energy, momentum and charge of the system which are defined by Nöther method is give as,

$$\begin{aligned} E &= \int d^3x T^{00} = \int d^3x \bar{\psi}(i\gamma^0\partial^0)\psi = \int d^3p \sum_r \omega(\vec{p})(N_r(\vec{p}) + \bar{N}_r(\vec{p}) + \text{const}) && (13.9) \\ \vec{P} &= \int d^3x T^{0i} = -\int d^3x \bar{\psi}(i\gamma^0\vec{\nabla})\psi = \int d^3p \sum_r \vec{p}(N_r(\vec{p}) + \bar{N}_r(\vec{p})) && (13.10) \\ Q &= q\int d^3x \psi^\dagger\psi = q\int d^3p \sum_r (N_r(\vec{p}) - \bar{N}_r(\vec{p})) && (13.11) \end{aligned}$$

It shows that the fermion (resp. anti-fermion) created by $c_r^\dagger(\vec{p})$ (resp. $d_r^\dagger(\vec{p})$) has energy $\omega(\vec{p})$, momentum $\vec{p}$ and charge q (resp. $-q$).

The derivation of these formulae (from third to fourth term) is not so straightforward. We need use formulae,

$$\bar{u}_r(\vec{p})\gamma^0 u_s(\vec{p}) = 2\omega(\vec{p})\delta_{rs}, \quad \bar{v}_r(\vec{p})\gamma^0 v_s(\vec{p}) = 2\omega(\vec{p})\delta_{rs}, \quad \bar{u}_r(\vec{p})\gamma^0 v_s(-\vec{p}) = 0 \tag{13.12}$$

which are proved by inserting the explicit form of the wave functions u, v.

Covariant anti-commutator As in the scalar field, we divide the fermion field into creation and annihilation part:

$$\begin{aligned} \psi(x) &= \psi^{(+)}(x) + \psi^{(-)}(x) \\ \psi^{(+)}(x) &= \int d^3x \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} \sum_r c_r(\vec{p})u_r(\vec{p})e^{-ipx} \\ \psi^{(-)}(x) &= \int d^3x \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} \sum_r d_r^\dagger(\vec{p})v_r(\vec{p})e^{ipx} && (13.13) \\ \bar{\psi}(x) &= \bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x) \\ \bar{\psi}^{(-)}(x) &= \int d^3x \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} \sum_r c_r^\dagger(\vec{p})\bar{u}_r(\vec{p})e^{ipx} \\ \bar{\psi}^{(+)}(x) &= \int d^3x \frac{1}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} \sum_r d_r(\vec{p})\bar{v}_r(\vec{p})e^{-ipx} && (13.14) \end{aligned}$$

The covariant anti-commutator is written in the form:

$$\begin{aligned} \left\{\psi_\alpha^{(\pm)}(x), \bar{\psi}_\beta^{(\mp)}(y)\right\} &= S^\pm(x-y), \quad S^\pm(x) = (i\gamma^\mu\partial_\mu + m)i\Delta^\pm(x) \\ \left\{\psi_\alpha(x), \bar{\psi}_\beta(y)\right\} &= S(x-y), \quad S(x) = (i\gamma^\mu\partial_\mu + m)i\Delta(x) && (13.15) \end{aligned}$$

To prove them, we plug the mode expansion into the anti-commutator and use identities such as $\sum_r u_r(\vec{p})_\alpha u_r(\vec{q})_\beta = (\gamma^\mu p_\mu + m)_{\alpha\beta}$.

Eq.(13.15) shows that fermion fields satisfies the causality since $\Delta(x) = 0$ for $x^2 < 0$.

Feynman propagator We have to be careful in the sign when we define the time ordering for fermions,

$$\begin{aligned} S_F(x-y) &= \langle 0|T(\psi_\alpha(x)\bar{\psi}_\beta(y))|0\rangle \\ &= \theta(x^0-y^0)\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle - \theta(y^0-x^0)\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle \\ &= \theta(x^0-y^0)S^{(+)}_{\alpha\beta}(x-y) - \theta(y^0-x^0)S^{(-)}_{\alpha\beta}(y-x) = \cdots \\ &= (i\gamma^\mu\partial_\mu + m)_{\alpha\beta}D_F(x-y) \\ &= \int \frac{d^4x}{(2\pi)^4} e^{-ip(x-y)} \frac{i(\gamma^\mu p_\mu + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} \end{aligned} \tag{13.16}$$

14 Quantization of Gauge (Maxwell) Field

Since gauge field A_μ has the gauge symmetry $\delta A_\mu = \partial_\mu \epsilon$, A_μ has *non-physical* components. It causes a problem in the canonical quantization.

We start from the Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)\,. \tag{14.1}$$

The canonical momentum for A_μ:

$$\pi_0 = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} = 0 \quad (!) \tag{14.2}$$

$$\pi_i = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i)} = F_{0i} = E_i. \tag{14.3}$$

While we would like to impose the canonical commutation relation

$$[A_\mu(\vec{x}), \pi_\nu(\vec{y})] = -i\eta_{\mu\nu}\delta(\vec{x}-\vec{y}) \tag{14.4}$$

it does not make sense since (14.2).

There are two approaches to define the quantization for such system.

1. Quantization of constrained system: this is a method developed by Dirac to quantize the constrained system (some canonical variables vanish)

2. Path integral: this is the most standard approach and will be explained in QFT2 course.

In this course, we take a short cut to save time. The origin of the problem is the gauge symmetry. This may be removed by using *gauge fixing condition*. Useful choices are,

$$\partial_\mu A^\mu = 0 \qquad \text{(Lorentz gauge)} \tag{14.5}$$
$$\vec{\nabla} \cdot \vec{A} = 0 \qquad \text{(Coulomb gauge)} \tag{14.6}$$

Here we pick the first one.

$$\begin{aligned} S &= -\frac{1}{4}\int d^4x \, (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\ &= -\frac{1}{2}\int d^4x \, (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) \\ &= -\frac{1}{2}\int d^4x \, (\partial_\mu A_\nu \partial^\mu A^\nu - \partial^\nu A_\nu \partial_\mu A^\mu) \end{aligned}$$

To obtain the third line from the second, we use the integration by parts for the second term. Assuming that the gauge fixing condition may be used at the level of Lagrangian, the second term in the third line vanishes. So one may use,

$$\mathcal{L}' = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \tag{14.7}$$

as the effective Lagrangian where the gauge symmetry is lost. The canonical momentum is obtained as

$$\pi'_\mu = \frac{\partial \mathcal{L}'}{\partial(\partial_0 A_0)} = -\partial_0 A_\mu \,, \tag{14.8}$$

and the commutation relation is obtained as,

$$[A_\mu(\vec{x}), \pi_\nu(\vec{y})] = -\left[A_\mu(\vec{x}), \dot{A}_\nu(\vec{y})\right] = i\eta_{\mu\nu}\delta(\vec{x}-\vec{y}) \,. \tag{14.9}$$

These are the same as the commutation relations of scalar fields while the sign is flipped for A^0.

As in scalar or Dirac fields, we may introduce the mode expansion,

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{\omega(\vec{p})}} \left\{ \sum_{r=0}^{3} \varepsilon_r^\mu(\vec{p}) \left(a_r(\vec{p}) e^{-ipx} + a_r^\dagger(\vec{p}) e^{ipx} \right) \right\} \tag{14.10}$$

Here $\varepsilon_r^\mu(\vec{p})$ is the polarization vector. We use the following convention for them,

$$\varepsilon_0^\mu(\vec{p}) = (1,0,0,0)^t \quad \text{(time component)} \tag{14.11}$$
$$\varepsilon_3^\mu(\vec{p}) = (0, \frac{\vec{p}}{|\vec{p}|})^t \quad \text{(longitudinal component)} \tag{14.12}$$
$$\epsilon_r^\mu(\vec{p}) = (0, \vec{\epsilon}_r)^t \quad \text{for } r = 1,2 \text{ (transverse components)} \tag{14.13}$$

$\vec{\epsilon}_r \in \mathbb{R}^3$ are chosen such that $\vec{\epsilon}_r \cdot \vec{\epsilon}_s = \delta_{rs}$, $\vec{\epsilon}_r \cdot \vec{p} = 0$.

It can be shown that the CCR for A_μ (14.9) is equivalent to the commutation relations for the creation/annihilation operators,

$$\left[a_r(\vec{p}), a_s^\dagger(\vec{q})\right] = -\eta_{rs}\delta(\vec{p}-\vec{q}) \tag{14.14}$$

and all the other brackets vanish. The covariant commutation relation,

$$[A^\mu(x), A^\nu(y)] = -i\eta^{\mu\nu}\delta(\vec{p}-\vec{q}), \quad i\Delta(x) = \frac{1}{(2\pi)^3}\int d^4p\delta(p^2)\varepsilon(p^0)e^{-ipx} \tag{14.15}$$

and the Feynman propagator,

$$D_F^{\mu\nu}(x-y) \;\;=\;\; \langle 0|T(A^\mu(x)A^\nu(y))|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-i\eta^{\mu\nu}}{p^2+i\epsilon}e^{-ip(x-y)} \tag{14.16}$$

are the same as those for scalar fields up to the sign.

The construction so far after the gauge fixing seems simple. It has, however, a serious problem – the positivity of the Hilbert space. It is natural that the n-particle states are spanned by the basis,

$$a_{r_1}^\dagger(\vec{p}_1)\cdots a_{r_n}^\dagger(\vec{p}_n)|0\rangle. \tag{14.17}$$

We can easily see that the time-like components $a_0^\dagger(\vec{p})|0\rangle$ has the negative inner-product,

$$\langle 0|a_0(\vec{p})a_0^\dagger(\vec{q})|0\rangle = -\delta(\vec{p}-\vec{q}).$$

It contradicts with the probability interpretation of the quantum mechanics.

The other problem is the positivity of the energy eigenvalue. The Hamiltonian of the system can be derived from Noether's theorem,

$$H = \int d^3p\sum_{r=0}^{3}\zeta_r\omega(\vec{p})a_r^\dagger(\vec{p})a_r(\vec{p}) + (\text{const.}), \qquad \zeta_r := -\eta_{rr}\,. \tag{14.18}$$

Suppose we neglect the infinite constant coming from normal ordering, the energy associated with time-like photon created by $a_0^\dagger(\vec{p})$ is negative.

Such problems are cured if we impose the gauge fixing condition (14.5) properly. We note first that to use $\partial_\mu A^\mu = 0$ as the operator relation is impossible since their covariant commutator,

$$[\partial_\mu A^\mu(x), A^\nu(y)] = -i\partial^\nu\Delta(x-y) \neq 0, \tag{14.19}$$

does not vanish. Therefore we impose such condition on the *Hilbert space*. We decompose the constraint into the annihilation and creation parts,

$$\partial_\mu A^\nu = (\partial_\mu A^\mu)^+ + (\partial_\mu A^\mu)^- \tag{14.20}$$

and use it as the restriction to the *physical* Hilbert space $\mathcal{H}_{\rm phys} \subset \mathcal{H}$,

$$\begin{aligned} |{\rm phys}\rangle \in \mathcal{H}_{\rm phys} &\leftrightarrow (\partial_\mu A^\mu)^+|{\rm phys}\rangle = 0 \quad (14.21)\\ \langle{\rm phys}| \in \mathcal{H}_{\rm phys}^\perp &\leftrightarrow (\partial_\mu A^\mu)^+|{\rm phys}\rangle = 0 \quad (14.22)\end{aligned}$$

We proceed to study what kind of states appear in $\mathcal{H}_{\rm phys}$. We observe,

$$p_\mu \varepsilon_r^\mu(\vec{p}) = \begin{cases} |\vec{p}| & r=0 \quad \text{time mode} \\ 0 & r=1,2 \quad \text{transverse mode} \\ -|\vec{p}| & r=3 \quad \text{longitudinal mode}\end{cases} \quad (14.23)$$

It implies,

$$(\partial_\mu A^\mu)^+|{\rm phys}\rangle = 0 \leftrightarrow (a_0(\vec{p}) - a_3(\vec{p}))|{\rm phys}\rangle = 0 \quad (14.24)$$

In the one particle state, the transverse modes $a_r^\dagger(\vec{p})|0\rangle$ $(r=1,2)$ is in the physical Hilbert space. The time and longitudinal modes $a_0^\dagger|0\rangle, a_3^\dagger|0\rangle$ does not satisfy the condition and are not physical. A combination of them, however,

$$(a_0^\dagger(\vec{p}) - a_3^\dagger(\vec{p}))|0\rangle \quad (14.25)$$

satisfies the physical state condition and therefore physical. This state, however, has zero-norm,

$$< 0|(a_0(\vec{p}) - a_3(\vec{p}))(a_0^\dagger(\vec{q}) - a_3^\dagger(\vec{q}))|0\rangle = -\delta(\vec{p}-\vec{q}) + \delta(\vec{p}-\vec{q}) = 0. \quad (14.26)$$

While the physical Hilbert space is generated by, (a) the creation operator of transverse mode: $a_{1,2}^\dagger(\vec{p})$ (b) a combination of time and longitudinal mode: $a_-^\dagger(\vec{p}) := a_0^\dagger(\vec{p}) - a_3^\dagger(\vec{p})$ the latter does not contribute to the observation since they have zero-norm.

To summarize the section, the description of the Maxwell field is given by the propagator (14.16) and physical Hilbert space generated by transverse modes $a_{1,2}^\dagger(\vec{p})$.

15 Interaction picture and perturbation theory

In quantum mechanics, we learned two pictures to describe the same system:

- **Schrödinger picture:** the state has time evolution,

$$i\frac{\partial \psi_S}{\partial t} = \hat{H}\psi_S\,. \quad (15.1)$$

Time evolution of the state is described by a unitary operator:

$$\begin{aligned}\psi_S(t) &= \hat{U}_S(t,t_0)\psi_S(t_0) \quad (15.2)\\ \hat{U}_S(t,t_0) &= e^{-i\hat{H}(t-t_0)} \quad (15.3)\end{aligned}$$

On the other hand, the operators (such as $\hat{x}, \hat{p}$) do not change in time: $\partial_t \mathcal{O}_S = 0$.

- Heisenberg picture: The operators evolve in time:

$$i\partial_t \mathcal{O}_H = \left[\mathcal{O}_H(t), \hat{H}\right] \tag{15.4}$$

It may be solved as,

$$\mathcal{O}_H(t)\hat{U}_S(t,t_0)^\dagger \mathcal{O}_S(t_0)U_S(t,t_0)\,. \tag{15.5}$$

On the other hand the state is static, $\psi_H(t) = \psi_S(t_0)$.

These two pictures are equivalent,

$$_S\langle B,t|\mathcal{O}_S|A,t\rangle = {}_S\langle B,t_0|U_S^\dagger(t,t_0)\mathcal{O}_S U_S(t,t_0)|A,t_0\rangle = {}_H\langle B|\mathcal{O}_H(t)|A\rangle_H\,. \tag{15.6}$$

Interaction picture In the perturbation theory, the Hamiltonian may be decomposed into two parts:

$$H = H_0 + H_I \tag{15.7}$$

where H_0 and H_I describes the free motion and the interaction (perturbation) respectively. Through such decomposition, one may introduce the third picture which will be particularly useful in quantum field theory which is refered to as the interaction picture. In this case,

1. Operators : time evolution by free Hamiltonian H_0
2. States : time evolution by interaction H_I

More precisely the Hamiltonian equation is written as,

$$i\partial_t \mathcal{O}_I(t) = [\mathcal{O}_I(t), H_0]\,, \tag{15.8}$$

$$i\partial_t \psi_I(t) = H_I(t)\psi_I(t), \tag{15.9}$$

where the interaction part $H_I(t)$ should be evolved in time by H_0;

$$H_I(t) = e^{iH_0(t-t_0)}H_I(t_0)e^{-iH_0(t-t_0)} \tag{15.10}$$

The time dependence of the state in the interaction picture can be solved as,

$$\psi_I(t) = U_I(t)\psi_I(t), \quad U_I(t) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\,, \tag{15.11}$$

which may be proved as,

$$\begin{aligned}
\partial_t\psi_I &= e^{iH_0(t-t_0)}(iH_0 - iH)e^{-iH(t-t_0)}\psi_H \\
&= e^{iH_0(t-t_0)}(-iH_I)e^{-iH(t-t_0)}\psi_H \\
&= e^{iH_0(t-t_0)}(-iH_I)e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\psi_H \\
&= -iH_I(t)\psi_I(t)
\end{aligned} \tag{15.12}$$

The interaction picture fits with the quantum field theory in the following reason. Firstly, the mode expansion of the fields include the time dependence according to the free Hamiltonian motion. They takes the relativistic invariant form as we have seen. While the interaction Hamiltonian $H_I(t)$ looks complicated, what we need to do is to replace the quantum field included in the interaction by the time dependent one (namely the mode expansion itself). Thus the complicated part seems to be in the description of the time evolution of the states. It turns out that they are simple as we see in the following.

Dyson expansion of S-matrix The time evolution from $t = -\infty$ to $t = \infty$ in the interaction picture gives the S-matrix:

$$\psi_I(t = \infty) = S\psi_I(t = -\infty) \tag{15.13}$$

namely, $S = U_I(\infty, -\infty)$. The time dependence of $U_I(t, t_0)$ is described by the differential equation with the initial value,

$$\begin{aligned} i\partial_t U_I(t, t_0) &= H_I(t)U_I(t, t_0) && (15.14)\\ U_I(t_0, t_0) &= 1\,. && (15.15) \end{aligned}$$

It can be solved by dividing the time interval into the infinitesimal pieeces, $\Delta t = \frac{t-t_0}{N}$ $(N >> 1)$, $t_\ell := t_0 + \ell\Delta t$. For such infinitesimal time evolution, one may use the approximation,

$$U_I(t_{\ell+1}, t_\ell) \approx 1 - i\Delta t H_I(t_i)\,. \tag{15.16}$$

The time evolution operator may be evaluated as,

$$\begin{aligned} U_I(t, t_0) &= \lim_{N\to\infty}(1 - i\Delta t H_I(t_{N-1}))(1 - i\Delta t H_I(t_{N-2}))\cdots(1 - i\Delta t H_I(t_0))\\ &= 1 - i\sum_{i=0}^{N-1} H_I(t_i)\Delta t + (-i)^2\sum_{i>j} H_I(t_i)H_I(t_j)(\Delta t)^2 + \cdots\\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int\cdots\int_{t_0}^{t} d^n t\, T(H_I(t_n)\cdots H_I(t_1)) && (15.17) \end{aligned}$$

In the last line, we convert the summation into integral. We note that the time ordering shows up naturally. Taking the limit $t \to \infty$, $t_0 \to -\infty$, we obtain an expression for S matrix,

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int\cdots\int_{-\infty}^{\infty} d^n t\, T(H_I(t_n)\cdots H_I(t_1)) \tag{15.18}$$

This expression for the S matrix is called Dyson expansion. We note that we need some modifications to define the S matrix which may be explained a bit later...

Application to QFT As explained, in the interaction picture, we replace operator by quantum fields including time and H_I by the interaction Lagrangian, $H_I \to -\int d^3x \mathcal{L}_I$. For example, the QED Lagrangian is written as,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \quad D_\mu = \partial_\mu - iqA_\mu \,, \tag{15.19}$$

the interaction Hamiltonian is $H_I = -q\int d^3x \bar{\psi}\gamma^\mu A_\mu \psi$.

The Dyson expansion for the QFT is written as,

$$S = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int_{(\mathbb{R}^4)^n} d^4x_1 \cdots d^4x_n \, T(\mathcal{L}_I(x_n) \cdots \mathcal{L}_I(x_1)) \tag{15.20}$$

Scattering amplitude We use the S-matrix and the external states to give the scattering amplitudes. Schematically the scattering of two particles with momentum $\vec{k}_1, \vec{k}_2$ into ℓ particles $\vec{p}_1, \cdots \vec{p}_\ell$ may be written as,

$${}_{t=\infty}\langle \vec{p}_1, \cdots, \vec{p}_\ell | \vec{k}_1, \vec{k}_2 \rangle_{t=-\infty} = \langle \vec{p}_1, \cdots, \vec{p}_\ell | S | \vec{k}_1, \vec{k}_2 \rangle \tag{15.21}$$

We normalize the external states (which appear in bra and ket) as,

- scalar field:

$$|\vec{p}\rangle = (2\pi)^{3/2}\sqrt{2\omega(\vec{p})}a^\dagger(\vec{p})|0\rangle \tag{15.22}$$

$$\langle\vec{p}| = (2\pi)^{3/2}\sqrt{2\omega(\vec{p})}a^\dagger(\vec{p})\langle 0| \tag{15.23}$$

- fermion:($\pm$ means particle/anti-particle)

$$|\vec{p}, r\rangle_+ = (2\pi)^{3/2}\sqrt{2\omega(\vec{p})}c_r^\dagger(\vec{p})|0\rangle \tag{15.24}$$

$$|\vec{p}, r\rangle_- = (2\pi)^{3/2}\sqrt{2\omega(\vec{p})}d_r^\dagger(\vec{p})|0\rangle \tag{15.25}$$

- photon: ($r = (1, 2)$ means the polarization, two transverse modes)

$$|\vec{p}, r\rangle = (2\pi)^{3/2}\sqrt{2\omega(\vec{p})}a_r^\dagger(\vec{p})|0\rangle \tag{15.26}$$

These normalizations come from the relativistic invariant measure for the momentum:

$$\int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{2\omega(\vec{p})} = \int \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0) \tag{15.27}$$

It simplifies the *wave functions* associated with the external fields:

- scalar field:

$$\langle 0|\phi(x)|\vec{p}\rangle = \int \frac{d^3q}{(2\pi)^{3/2}\sqrt{2\omega(\vec{p})}} (2\pi)^{3/2}\sqrt{2\omega(\vec{p})}a^\dagger(\vec{p})|0\rangle = e^{-ipx} \tag{15.28}$$

- Dirac field:

$$\langle 0|\psi(x)|\vec{p},r\rangle_+ = u_r(\vec{p})e^{-ipx} \tag{15.29}$$

$$\langle 0|\bar{\psi}(x)|\vec{p},r\rangle_- = \bar{v}_r(\vec{p})e^{-ipx} \tag{15.30}$$

- photon:

$$\langle 0|A_\mu(x)|\vec{p},r\rangle = \varepsilon^\mu_r(\vec{p})e^{-ipx} \tag{15.31}$$

16 Feynman rule

We have seen so far that the scattering amplitude is written in the form $\langle\vec{p}_1,\cdots,\vec{p}_\ell|S|\vec{k}_1,\vec{k}_2\rangle$ where S has the Dyson expansion (15.20) and the state $|\vec{k}_1,\vec{k}_2\rangle\sim a^\dagger(k_1)a^\dagger(k_2)|0\rangle$ up to the normalization factor.

This is in principle doable already since $\mathcal{L}_I$ and the state are described by free oscillators. Here we make it more explicit. What we need to do is following,

(i) We change the order of operators in $T(\mathcal{L}_I(x_n)\cdots\mathcal{L}_I(x_1))$, namely move the creation (annihilation) operator to the left (right) by using the commutation relations. Such manipulation is called "normal ordering".

(ii) operate the annhilation (creation) operators to the ket (bra) state. The operators which have exactly the same number of of annihilation (creation) operators as the number of particles in the initial (final) state can have nonvanishing inner product.

Normal ordering The change of ordering (creation to left, annhilation to right) is called normal ordering. We illustrate it by the product of two operators,

$$A=A^++A^-,\quad B=B^++B^- \tag{16.1}$$

(operator with $+$ $(-)$ index is annihilation (creation) part). We write $N(AB)$ for the normal ordered product for AB. It is written explicitly as,

$$N(AB)=A^+B^++A^-B^++(-1)^{|A||B|}B^-A^++A^-B^-\,. \tag{16.2}$$

It is different from the product by the commutator $[A^+B^-]$ which can expressed as the vacuum expectation value,

$$AB=N(AB)+\langle 0|AB|0\rangle. \tag{16.3}$$

T-product, normal ordering and Feynman propagator By combining the T-product,

$$T(A(x)B(y))=\theta(x^0-y^0)A(x)B(y)+(-1)^{|A||B|}\theta(y^0-x^0)B(y)A(x), \tag{16.4}$$

with the normal product, the previous result (16.3) is replaced by,

$$T(A(x)B(y))=N(A(x)B(y))+\langle 0|T(A(x)B(y))|0\rangle. \tag{16.5}$$

The last term on the right hand side is the Feynman propagator. For the scalar case, for example,

$$\langle 0|T(\phi(x)\phi(y))|0\rangle=D_F(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)} \tag{16.6}$$

Propagator for the other fields are,

$$\text{fermion} \quad \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \qquad \text{photon} \quad \frac{-i\eta^{\mu\nu}}{p^2 + i\epsilon} \tag{16.7}$$

In the following we use the notation

$$\underbracket{A(x)B(y)} := \langle 0|T(A(x)B(y))|0\rangle \tag{16.8}$$

The connected operators are replaced by the propagator. It is referred to as the *contraction*.

Wick's theorem One may generalize (16.5) for the product of time-ordered n operators:

$$\begin{aligned} T(A_1 \cdots A_n) &= N(A_1 \cdots A_n) \\ &+ \sum_{i<j} N(A_1 \cdots \underbracket{A_i \cdots A_j} \cdots A_n) \\ &+ \sum_{i<j<k<l} \Big(N(A_1 \cdots \underbracket{A_i \cdots A_j} \cdots \underbracket{A_k \cdots A_l} \cdots A_n) + ((ik)(jl) + (il)(jk)) \Big) \\ &+ \cdots . \end{aligned} \tag{16.9}$$

On the right hand side, we take contractions for all possible pairs in the product and replace them by propagators. This formula is referred to as Wick's theorem.

To prove it, we use the induction. For $n = 2$, we have already proved it in (16.5).

Feynman diagram for scalar field As the simplest example of the interacting theory, we consider the ϕ^4 interaction for scalar field,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \quad \mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi)^2 - m^2\phi^2\right), \quad \mathcal{L}_I = -\frac{\lambda}{4!}\phi^4 . \tag{16.10}$$

n-th order term in Dyson expansion of the scattering amplitude consists of

$$\langle \vec{p}_1, \cdots, \vec{p}| \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n T(\mathcal{L}_I(x_1) \cdots \mathcal{L}_I(x_n))|\vec{k}_1\vec{k}_2\rangle.$$

We apply Wick's theorem to $T(\mathcal{L}_I(x_1)\cdots\mathcal{L}_I(x_n))$. Each contraction is replaced by propagator. The remaining (uncontracted) fields remains in the inner product. We need to have two annihilation operators and ℓ creation operators to make such inner product nonvanishing. Such procedure is carried out by using a graphic rule,

1. internal line (contraction of two ϕs in $\mathcal{L}_I$). It is replaced by the propagator $D_F(x_1 - x_2)$.

2. external line: evaluation of inner product such as

$$\langle \vec{p}_1, \cdots, \vec{p}_\ell| \underbrace{\phi^- \cdots \phi^-}_{\ell} \phi^+\phi^+|\vec{k}_1\vec{k}_2\rangle$$

which is evaluated as products of factors of the wave functions,

$$\langle 0|\phi(x)|\vec{k}\rangle = e^{-ikx}, \quad \langle \vec{p}|\phi(x)|0\rangle = e^{ipx} \tag{16.11}$$

Such contraction with bra and ket states is called as "internal line".

3. vertex: Each term in $\mathcal{L}$ has an integration of the form,

$$-i\lambda \int d^4x, \tag{16.12}$$

which should be performed after the contractions (internal and external lines). The factor $\frac{1}{4!}$ in the interaction Lagrangian will be (mostly) cancelled by the combinatorics factor (which ϕ should be contracted with other ϕ).

4. Symmetry factor $1/|G|$: when the diagram has symmetry (say finite group G), we need divide the factor by the order of the goup $|G|$.

This is called as the Feynman rule in the coordinate space.

It is, however, simpler to use the expression for the propagator,

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}.$$

The x dependence of the diagram comes from the propagator and the wave function. It is given in the form $e^{-i(p_1+\cdots+p_4)x}$ for each vertex. It is integrated for each vertex and it gives $\delta^{(4)}(p_1+\cdots+p_4)$. Since the x integrations are replaced by the momentum integration, the Feynman rule is rewritten as

1. internal line: we assign the factor $\frac{i}{p^2-m^2+i\epsilon}$
2. external line: we assign 1. (e^{ipx} factor is integrated.)
3. vertex: $-i\lambda\delta(p_1+\cdots+p_4)$.
4. symmetry factor: $1/|G|$.
5. integration over the momentum assigned to internal line.

The number of the integration of momentum was originally I (number of internal lines). It is reduced by V. One delta function can not be cancelled since it gives the overall momentum integration, $\delta(\sum_{i\in\mathrm{OUT}} p_i - \sum_{i\in\mathrm{IN}} p_i)$. The number of effective momentum integration is $I - V + 1 = L$ where L is the number of loops in the diagram. Usually the delta function factor for the vertex is not written in the textbook because such integration is assumed.

Feynman rule for QED The Feynman rule for QED can be derived in a parallel fashion. The system consists of Dirac fermion with photon. The interaction is $q\bar{\psi}\gamma^\mu A_\mu \psi$. We go to the momentum representation directly.

1. internal line:

$$\text{fermion:} \quad \frac{i(p_\mu \gamma^\mu + m)}{p^2 - m^2 + i\epsilon}, \qquad \text{photon:} \quad \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} \tag{16.13}$$

2. external line:

$$\text{ingoing :} \qquad \text{fermion: } u_r(\vec{p}), \quad \text{anti-fermion: } \bar{v}_r(\vec{p}), \quad \text{photon: } \quad \epsilon_r^\mu(\vec{p}) \tag{16.14}$$

$$\text{outgoing :} \qquad \text{fermion: } \bar{u}_r(\vec{p}), \quad \text{anti-fermion: } v_r(\vec{p}), \quad \text{photon: } \quad \epsilon_r^{\mu *}(\vec{p}) \tag{16.15}$$

3. vertex: $iq\gamma^\mu$

4. Symmetry factor: $1/|G|$

5. momentum integration over loops